Random Navier-Stokes-type system for electrorheological fluid

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Joint work with V. Zhikov (Russia)

SPA — 2015,
Oxford, 17 July 2015

Motivation

The talk focuses on homogenization results for Navier–Stokes-type system describing electrorheological fluid with random characteristics.

Rheological properties of some fluids might change essentially in the presence of an electromagnetic field. For such fluids the viscous stress tensor is not only a nonlinear function of the deformation velocity tensor, it also depends on the spatial argument.

The mathematical models of electro-rheological fluids are presented in the book **M. Ruzicka, Electrorheological fluids: modeling and mathematical theory. '00.**

Generalized Naviers-Stokes equations

The corresponding system of equations takes the form (so called generalized Naviers-Stokes equations)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div}\big(A(x, Du)\big) + \mathrm{div}(u \otimes u) + \nabla\pi = 0, \quad 0 \le t \le T, \\ \mathrm{div} u = 0, \qquad u|_{\partial G} = 0, \qquad u|_{t=0} = u_0, \end{cases}$$

where tensor $A(x, \xi)$ satisfies $p(\cdot)$-growth condition with

$$1 < \alpha \le p(x) \le \beta < \infty.$$

Sobolev spaces with variable exponent. Existing results

Lebesgue and Sobolev spaces with variable exponents. J.Rakocnik, O.Kovacik; X.Fan; S.Samko; L.Diening, M. Ruzicka, P. Hasto, P. Harjulehto; V.Zhikov. '89 — '93

Lavrentiev phenomenon in Sobolev spaces with variable exponents. V.Zhikov '91

Degenerated elliptic and parabolic equations in Variable Sobolev spaces. S.Antontsev, S.Shmarev '04, '06

Variational problems in variable Sobolev spaces, G.Mignione, A.Acerbi '01; A.Braides, A.Defrenceschi '98; etc.

Homogenization in variable Sobolev spaces S.Kozlov '89; V.Zhikov '90; B.Amaziane, S. Antontsev, L.Pankratov, A.P. '08

Homogenization problem

We assume that viscous stress tensor is a rapidly oscillating random statistically homogeneous function of x, that is

$$A_\varepsilon(x,\xi) = A\Big(\frac{x}{\varepsilon},\xi\Big),$$

where $A(y,\xi) = A(y,\xi,\omega)$ is a statistically homogeneous (stationary) ergodic tensor filed in y, and ε being a small positive parameter that characterizes the microscopic length scale in the fluid.

Two-scale convergence. Some references

Periodic two-scale convergence. G.Nguetseng '89; G.Allaire, '91

Stochastic two-scale convergence in the mean. A.Bourgeat, A.Mikelic, S.Wright '94

Stochastic two-scale convergence. A.P., V.Zhikov '06; M.Heida '11

Homogenization problem

Given a bounded Lipschitz domain $G \subset \mathbb{R}^d$ we consider in the cylinder $Q_T = G \times (0, T)$ the following generalized Naviers-Stokes system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div}\big(A_\varepsilon(x, Du)\big) + \mathrm{div}(u \otimes u) + \nabla \pi = 0, \quad 0 \le t \le T, \\ \mathrm{div} u = 0, \qquad u|_{\partial G} = 0, \qquad u|_{t=0} = u_0, \end{cases}$$

here u is the velocity vector, $D = Du$ is the symmetric part of the gradient, $\nabla \pi = \nabla_x \pi$, π is the pressure, $u \otimes u = \{u_i u_j\}$, and $\mathrm{div}(u \otimes u) = u_i \frac{\partial u_j}{\partial x_i}$.

We are aimed at studying the limit behaviour of its solutions as $\varepsilon \to 0$. This is so called homogenization problem.

Homogenization problem

Our goal is to construct a homogenized (effective) system of equations and to prove the convergence result. We will also study the properties of the homogenized system.

One of the crucial problems here is passage to the limit in the fluxes (the expressions under the divergence sign). Under certain assumptions on A, we characterize the limit of fluxes of the original ε-equations and show that this limit coincides with the flux of the homogenized system.

Probability space

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space.

We assume that the probability space is equipped with a measure preserving dynamical system T_y, $y \in \mathbb{Z}^d$. It possesses the following properties

- $T_{y+z} = T_x \circ T_y; \quad T_0 = Id.$
- $\mathbf{P}(T_x \mathcal{A}) = \mathbf{P}(\mathcal{A})$ for any $\mathcal{A} \in \mathcal{F}$ and any $x \in \mathbb{R}^d$.
- $T_\cdot \, : \, (\Omega \times \mathbb{R}^d) \mapsto \Omega$ is a measurable mapping; R^d being equipped with the Borel σ-algebra $\mathcal{B}$.

We also assume that dynamical system T_x is ergodic, that is any function which is invariant w.r.t. T_x, $x \in \mathbb{R}^d$ is equal to a constant almost surely (a.s.).

Denote by ∂_ω^j the generator of T_y in the j-th coordinate direction, and by $\mathcal{D}^j$ its domain. It is known that $\mathcal{D} = \bigcap\limits_{j=1}^{d} \mathcal{D}^j$ is dense in $L^2(\Omega)$, and that $i\partial_\omega^j$ is a self-adjoint operator in $L^2(\Omega)$.

We say that a function $\varphi \in L^2(\Omega)$ belongs to $\mathcal{D}^\infty(\Omega)$ if for any $j_1,\, j_2\, \dots,\, j_N$ we have $\partial_\omega^{j_1} \dots \partial_\omega^{j_N} \varphi \in L^2(\Omega)$.

It is known that $\mathcal{D}^\infty(\Omega)$ is dense in $L^2(\Omega)$.

Assumptions

We assume that

$$A(y, \xi, \omega) = \mathbf{A}(T_y \omega, \xi)$$

where $\mathbf{A}(\omega, \xi)$ is a symmetric $(d \times d)$-matrix, $\omega \in \Omega$, and ξ varies over the space of symmetric $(d \times d)$-matrices.

It is assumed that $\mathbf{A}(\omega, \xi)$ is a Carathéodory function that satisfies the following conditions

- $(\mathbf{A}(\omega, \xi_1) - \mathbf{A}(\omega, \xi_2)) \cdot (\xi_1 - \xi_2) > 0, \quad \xi_1 \neq \xi_2,$
- $\mathbf{A}(\omega, \xi) \cdot \xi \geq c_0 |\xi|^{p(\omega)} - 1, \quad c_0 > 0,$
- $|\mathbf{A}(\omega, \xi)|^{p'(\omega)} \leq c_1 |\xi|^{p(\omega)} + 1, \quad p'(\omega) = \frac{p(\omega)}{p(\omega) - 1}.$

The exponent $p(\omega)$ is measurable and satisfies the inequality

$$1 < \alpha \leq p(\omega) \leq \beta < \infty.$$

Functional spaces

We first introduce functional spaces:

$$J = \Big\{ u \in (C_0^\infty(G))^d, \, \mathrm{div}\, u = 0 \Big\},$$

H is the closure of the set J in $(L^2(G))^d$,

$X = X_{\varepsilon,\omega}$ is the closure of the set

$$\Big\{ u \in (C^\infty(\bar{Q}_T))^d, \quad \mathrm{div}_x u = 0, \quad u = 0 \text{ near } \partial G \times [0,T] \Big\}$$

in the Luxemburg norm

$$\|Du\|_{L^{p_\varepsilon(\cdot)}(Q_T)} = \inf \Big\{ \lambda > 0 \, : \, \int_{Q_T} |\lambda^{-1} Du|^{p_\varepsilon(x)} \, dxdt \le 1 \Big\}$$

with $p_\varepsilon(x) = p(T_{x/\varepsilon}\omega)$.

Definition. A function $u \in X \cap L^\infty(0,T,H)$ is a *weak solution* of generalized Navier-Stokes system, if

- for any $\eta \in J$, we have

$$\int_\Omega u \cdot \eta dx \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_\Omega [A_\varepsilon - (u \otimes u)] \cdot \nabla \eta dx dt = 0 \qquad \forall\, t_0,\, t_1 \ \in\ [0,T],$$

$$\lim_{t\to+0} \int_\Omega u \cdot \eta dx = \int_\Omega u_0 \cdot \eta dx;$$

- the energy inequality

$$\int_\Omega u \cdot u dx \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_\Omega A_\varepsilon \cdot Du\, dx dt \le 0$$

holds for almost all $t_0, t_1 \in [0,T]$.

Remark

Remark

In addition to the integral identity, the definition of a weak solution contains an energy inequality.

The integral identity allows one to conclude that $u(\cdot, t)$ is a weakly continuous function of $t \in [0, T]$ with values in H. However it does not yield the energy equality, which apparently does not hold. In other words, the theory admits the strict energy inequality, which means the violation of energy conservation law.

A critical point here is how the energy balance can be recovered.

Existence result

Theorem

Assume that

$$\alpha \geq \alpha_0(d) = \max\left\{\frac{d + \sqrt{3d^2 + 4d}}{d+2}, \frac{3d}{d+2}\right\}$$

and

$$\alpha \leq p(x) \leq \beta < \infty.$$

Then generalized Navier-Stokes system has a weak solution for any $u_o \in H$.

In dimension 3 we have $\alpha_0(3) \in (1.84,\ 1.85)$.

The condition $p \geq \alpha$ ensures that the convective term $u \otimes u$ can be estimated in terms of the viscous term.

Lemma

If $u \in X \cap L^\infty(0, T, H)$, *then*

$$|u|^2 \in L^1(0, T, L^{\alpha'}(\Omega))$$

.

Notice that if $\alpha = \frac{3d+2}{d+2}$ (Ladyzhenskaya-Lions exponent) then

$$|u|^2 \in L^{\alpha'}(0, T, L^{\alpha'}(\Omega)) = L^{\alpha'}(Q_T).$$

In this case the convective term is *completely* subjected to viscous one.

Homogenization

In order to determine the homogenized tensor we introduce the following spaces:

- $L^{p(\cdot)}(\Omega)$;
- $G(\Omega)$ is the closure of the set $\{D\varphi \,:\, \varphi \in (\mathcal{D}^\infty(\Omega))^d, \mathrm{div}_\omega \varphi = 0\}$ in $L^{p(\cdot)}(\Omega)$; here for a random vector $f \in (\mathcal{D}(\Omega))^d$ we define

$$\mathrm{div}_\omega f = \sum_j \partial_\omega^j f^j.$$

- $G(\Omega)^\perp$ is the subspace of functions $b \in L^{p'(\cdot)}(\Omega)$ such that

$$\int_\Omega b \cdot v \, d\mathbf{P} = 0 \qquad \text{for any } v \in G(\Omega).$$

"Cell" problem

The basic auxiliary problem in Ω reads

$$\text{find } v \in G(\Omega) \ \text{ such that } \quad \int_{\Omega} \mathbf{A}(\omega, \xi + v) \cdot z(\omega)\, d\mathbf{P} = 0, \quad \forall z \in G(\Omega),$$

here ξ is a constant matrix.

Lemma

The above problem has a unique solution for any symmetric matrix $\xi \in \mathbb{R}^{d(d+1)/2}$.

We denote this solution by v_ξ

Homogenized tensor

The homogenized tensor is now defined by

$$A^{\text{eff}}(\xi) = \int_\Omega \mathbf{A}(\omega, \xi + v_\xi)\, d\mathbf{P}.$$

Lemma

The homogenized tensor A^{eff} is strictly monotone and continuous.

Homogenized tensor

In order to characterize the coerciveness properties of A^{eff} we introduce the following functionals

$$\mathbf{f}(\xi) = \min_{v\in G(\Omega)} \int_\Omega \frac{|\xi + v|^{p(\omega)}}{p(\omega)} d\mathbf{P}.$$

and its conjugate (in the sense of Young)

$$\mathbf{f}^*(\xi) = \min \int_\Omega \frac{|z|^{p'(\omega)}}{p'(\omega)} d\mathbf{P},$$

where the minimum is taken over all $z \in G^\perp(\Omega)$ such that $\int_\Omega z\, d\mathbf{P} = \xi$. The functional $\mathbf{f}$ and $\mathbf{f}^*$ are convex, even, and positive for $\xi \neq 0$.

Lemma

There exist $C_1 > 0$ and $C_2 > 0$ such that

$$A^{\text{eff}}(\xi)\cdot\xi \geq C_1\mathbf{f}(\xi) - 1, \qquad \mathbf{f}^*(A^{\text{eff}}(\xi)) \leq C_2\mathbf{f}(\xi) + 1.$$

Orlicz spaces

We introduce the following Orlicz space $L^{\mathbf{f}}(Q_T)$

$$L^{\mathbf{f}}(Q_T) = \{v \in (L^1(Q_T))^{d(d+1)/2} \ : \ \int_{Q_T} \mathbf{f}(v)\, dxdt < \infty\}$$

with the Luxemburg norm

$$\|v\|_{L^{\mathbf{f}}} = \inf \Big\{\theta > 0 \ : \ \int_{Q_T} \mathbf{f}(\theta^{-1}v)\, dxdt \leq 1\Big\},$$

and Sobolev-Orlicz spaces

$$W_0^{\mathbf{f}}(G) = \{u \in (W_0^{1,1}(G))^d,\ \mathrm{div}\, u = 0,\ \mathbf{f}(Du) \in L^1(G)\},$$
$$X^{\mathbf{f}}(Q_T) = \{u \in L^1\big(0,T; (W_0^{1,1}(G))^d\big),\ \mathrm{div}_x u = 0,\ \mathbf{f}(Du) \in L^1(Q_T)\}.$$

The spaces $W_0^{\mathbf{f}}(G)$ and $X^{\mathbf{f}}(Q_T)$ are equipped with the norms $\|Du\|_{L^{\mathbf{f}}(G)}$ and $\|Du\|_{L^{\mathbf{f}}(Q_T)}$, respectively.

Lemma

The set $C_{0,\mathrm{sol}}^{\infty}(G)$ *is dense in* $W_0^{\mathbf{f}}(G)$, *the set*

$$\{u \in C^{\infty}([0,T]; C_0^{\infty}(G)) \,:\, \operatorname{div} u = 0\}$$

is dense in $X^{\mathbf{f}}(Q_T)$.

We formally write down the homogenized equations as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(A^{\mathrm{eff}}(Du)\big) + \operatorname{div}(u \otimes u) + \nabla \pi = 0, \quad 0 \le t \le T, \\ \operatorname{div} u = 0, \qquad u|_{\partial G} = 0, \qquad u|_{t=0} = u_0, \end{cases}$$

Definition of a solution

Definition. A function $u \in X^{\mathbf{f}}(Q_T) \cap L^\infty(0,T,H)$ is a *weak solution* of the homogenized problem, if

- for any $\eta \in J$, we have

$$\int_\Omega u\cdot\eta dx\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\int_\Omega [A^{\text{eff}}(Du) - (u\otimes u)]\cdot\nabla\eta dxdt = 0 \qquad \forall\, t_0,\ t_1 \in [0,T],$$

$$\lim_{t\to+0}\int_\Omega u\cdot\eta dx = \int_\Omega u_0\cdot\eta dx;$$

- the energy inequality

$$\int_\Omega u\cdot u dx\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\int_\Omega A^{\text{eff}}(Du)\cdot Du\, dxdt \le 0$$

holds for almost all $t_0, t_1 \in [0,T]$.

Homogenization theorem

Theorem

Suppose that $\alpha \geq \alpha_0(d)$ *and*

$$\beta < \alpha^* = \begin{cases} \frac{\alpha d}{d-\alpha}, & \text{if } \alpha < d \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, almost surely, a solution u_ε *of the original generalized Navier-Stokes system converges for a subsequence to a solution of the homogenized problem.*

Remark. In the homogenization theorem the choice of a converging subsequence and the corresponding solution of the homogenized problem might depend on ω.

Stochastic two-scale convergence

One of the key tools in proving the homogenization theorem is stochastic two-scale convergence. The stochastic two-scale convergence in the mean was introduces in Borgeat, Mikelic, Wright '98. We use another version of stochastic two-scale convergence.

Let $\{v^\varepsilon = v^\varepsilon(x,t,\widetilde{\omega}),\ 0<\varepsilon\le\varepsilon_0\}$ be a family of functions such that for $\mathbf{P}$ almost all $\widetilde{\omega}\in\Omega$ we have $v^\varepsilon(\cdot,\cdot,\widetilde{\omega})\in L^p(Q_T)$ for all $\varepsilon\in(0,\varepsilon_0]$.

Definition

We say that the family $v^\varepsilon\in L^p(Q_T)$ weakly stochastic two-scale converges, as $\varepsilon\to 0$, to a function $v=v(x,t,\omega)$, $v\in L^p(Q_T\times\Omega)$, if a.s.

$$\limsup_{\varepsilon\to 0}\|v_\varepsilon\|_{L^p(Q_T)}<\infty,$$

and for any $\varphi\in C_0^\infty(Q_T)\times\mathcal{D}^\infty(\Omega)$ it holds

$$\lim_{\varepsilon\to 0}\int_{Q_T} v^\varepsilon(x,t)\varphi^\varepsilon(x,t)\,dxdt \longrightarrow \int_{Q_T}\int_\Omega v(x,t,\omega)\varphi(x,t,\omega)\,dxdtd\mathbf{P},$$

where $\varphi^\varepsilon(x,t)=\varphi(x,t,\tau_{x/\varepsilon}\omega)$.

Two-scale convergence

Notice that the two-scale limit function might also depend on the realization of the medium $\widetilde{\omega}$.

Although the two-scale limit is defined separately for each typical realization of the medium, that is for a given $\widetilde{\omega}$, the limit function is defined on the whole Ω. We do not indicate the dependence on $\widetilde{\omega}$ explicitly.

Properties of two-scale convergence

Lemma

Every family of functions $\{v^\varepsilon,\ \varepsilon > 0\}$ *such that* $\|v^\varepsilon\|_{L^2(Q_T)} \le C$ *, weakly two-scale converges for a subsequence to some* $v = v(x,t,\omega)$*,* $v \in L^p(Q_T \times \Omega)$*.*

Lemma

Let a family v^ε *be such that a.s.*

$$\|v^\varepsilon\|_{L^p(Q_T)} \le C, \qquad \lim_{\varepsilon\to 0} \varepsilon\|\nabla_x v^\varepsilon\|_{L^p(Q_T)} = 0.$$

Then, for a subsequence,

$$v^\varepsilon \stackrel{2}{\rightharpoonup} v \qquad \text{weakly two-scale in } L^p(Q_T),$$

with $v = v(x,t)$*,* $v \in L^p(Q_T)$*.*

Lemma

Let a family v^ε satisfy a.s the estimate

$$\|v^\varepsilon\|_{L^p(Q_T)} + \|\nabla_x v^\varepsilon\|_{L^p(Q_T)} \le C$$

for all $\varepsilon \in (0, \varepsilon_0]$. Then, for a subsequence,

$$\nabla_x v^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_x v(x,t) + v_1(x,t,\omega) \qquad \text{weakly two-scale in } L^p(Q_T \times \Omega),$$

with $v = v(x,t)$, $v \in L^p((0,T); W^{1,p}(G))$ and $v_1 \in L^p(Q_T; L^p_{\rm pot}(\Omega))$, where $L^p_{\rm pot}(\Omega)$ is the closure in $L^p(\Omega)$ of the set $\{\partial_\omega u \,:\, u \in \mathcal{D}^\infty(\Omega)\}$.

Lemma

For a subsequence,

$$u^\varepsilon \overset{2}{\rightharpoonup} u(x,t) \qquad \text{weakly two-scale in } L^\alpha(Q_T),$$

$$Du^\varepsilon \overset{2}{\rightharpoonup} Du(x,t) + u_1(x,t,\omega) \qquad \text{weakly two-scale in } L^\alpha(Q_T \times \Omega),$$

where $u_1(x,t,\cdot) \in \mathcal{G}(\Omega)$ *a.a. in* Q_T *and*

$$\int_{Q_T}\int_\Omega |Du(x,t) + u_1(x,t,\omega)|^{p(\omega)}\, dxdtd\mathbf{P}(\omega) < \infty;$$

$$A(\cdot/\varepsilon, Du^\varepsilon) \overset{2}{\rightharpoonup} z(x,t,\omega) \qquad \text{weakly two-scale in } L^{\beta'}(Q_T \times \Omega),$$

where

$$\int_{Q_T}\int_\Omega |z(x,t,\omega)|^{p'(\omega)}\, dxdtd\mathbf{P}(\omega) < \infty,$$

$z(x,t,\cdot) \in \mathcal{G}^\perp(\Omega)$ *a.a. in* Q_T*. Moreover,* $z_0(x,t) = \int_\Omega z(x,t,\omega)\, d\mathbf{P}(\omega)$.