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partially joint with
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(Warwick)

# Percolation

Physical phenomenon:

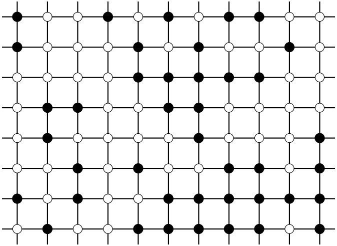
- Fluid through porous medium
- Material sciences, epidemics, networks

Main motivations:

- Phase transition
- Universality
- Challanges

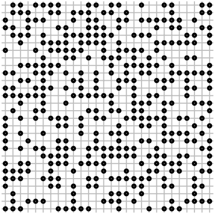
# Mathematical model

Broadbent, Hammersley '57

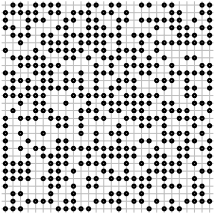


- Infinite graph $G = (V, E)$, e.g. $\mathbb{Z}^d$
- Parameter $p \in [0, 1]$
- Open each vertex $x \in V$ independently with probability $p$
- Connectivity properties of the induced subgraph

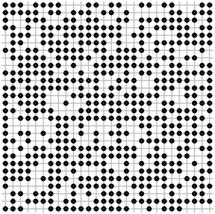
# Phase transition



$p = 0.500$



$p = 0.593$

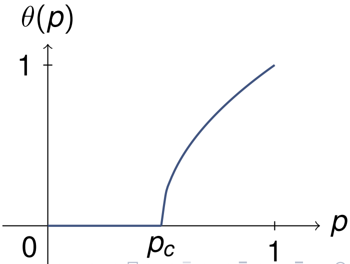


$p = 0.700$

Is the origin connected to infinity?

$$\theta(p) = \mathbb{P}[o \leftrightarrow \infty]$$

$$p_c(G) = \sup\{p; \theta(p) = 0\}$$



# On $\mathbb{Z}^d$

A lot is known:

- $d \geq 2 \quad \Rightarrow \quad$ non-trivial transition ($p_c \in (0, 1)$) not true for $d = 1$
- $d \geq 2 \quad \Rightarrow \quad \theta$ smooth for $p > p_c$
- $d \geq 11 \quad \Rightarrow \quad \theta$ continuous $+$ critical behavior
- $d = 2 \quad \Rightarrow \quad \theta$ continuous $+$ critical behaviour

Still a lot to learn:

- $\theta$ continuous for all $d$?
- critical behavior for all $d$?

# Other graphs

Is $p_c(G) \in (0, 1)$?

- If degree $G$ bounded by $\Delta \quad \Rightarrow \quad p_c \geq 1/\Delta$.
- What about $p_c(G) < 1$?

*"The first step in a study of percolation on other graphs $(\cdots)$ will be to prove that the critical probability on these graphs is smaller than one."*

Benjamini, Schramm

# Other graphs

We know $p_c(G) < 1$ for:

- Regular trees (strong symmetries)
- Expanders [Benjamini, Schramm]
- Cayley: with exponential growth [Lyons]
- Cayley: finitely generated, with one end [Babson, Benjamini]
- Cayley: intermediate growth [Muchnik, Pak]

Techniques range from

- Entropy vs energy
- Transport principle
- Analytical tools
- Homology...

# Isoperimetric inequalities

$p_c(\mathbb{Z}^1) = 1$, does dimension play a role?

On $\mathbb{R}^d$, every compact $A \subseteq \mathbb{Z}^d$ with smooth boundary:

$$|\partial A| \geq c_d |A|^{(d-1)/d} \qquad (\text{recall } |\partial B| = c_d |B|^{(d-1)/d})$$

## Definition

$$\dim(G) \geq d \quad \overset{\text{def}}{\Leftrightarrow} \quad \inf_{A \subseteq V;\text{ finite}} \frac{|\partial A|}{|A|^{(d-1)/d}} > 0$$

- Examples: $\mathbb{Z}^d$, Sirpinsky graphs, regular trees...
- Very useful concept to study random walks on $G$

# Benjamini-Schramm's question

## Question

Is it true that $\dim(G) > 1$ implies $p_c(G) < 1$?

Two results in this direction:

## Theorem (Kozma)

If $G$ is planar, with polynomial growth, no accumulation points then:

$$\dim(G) > 1 \quad \Rightarrow \quad p_c(G) < 1.$$

## Theorem (Candellero, T.)

If $G$ is transitive, with polynomial growth, then:

$$\dim(G) > 1 \quad \Rightarrow \quad p_c(G) < 1.$$

# Local isoperimetric inequalities

## Definition - local isoperimetric dimension

$$\dim_\ell(G) \geq d \quad \overset{\text{def}}{\Leftrightarrow} \quad \inf_{B=B(x,r)} \; \inf_{\substack{A \subseteq B; \\ |A| \leq |B|/2}} \frac{|\partial_B A|}{|A|^{(d-1)/d}} > 0$$

Counterexamples: regular trees, two $\mathbb{Z}^d$'s glued...

## Theorem (T.)

If $G$ has polynomial growth, then:

$$\dim_\ell(G) > 1 \;\; \Rightarrow \;\; p_c(G) < 1.$$

Moreover, $p_u(G) < 1$, dependent models, Ising [Häggström].

# Renormalization

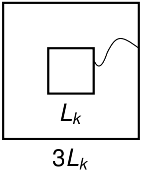
Multi-scale, in a nutshell:

- Coarse-graining procedure
- Process $\to$ dynamics
- Geometry aware
- Robust to changes
- Well adapted for perturbative systems
- Well adapted for $\mathbb{Z}^d$
- Various models: rwre, sand-piles, interface motion...

# Renormalization example

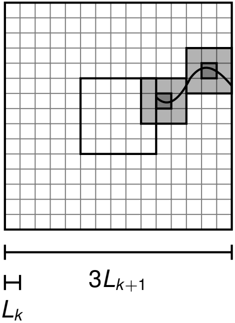
Let us show $p_c(\mathbb{Z}^2) > 0$:

$L_k = 5^k$, for $k \geq 0$

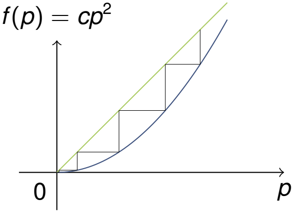
$A_k =$ 


$q_k(p) = \mathbb{P}[A_k]$

$A_{k+1} \Rightarrow A_k \cap A'_k$




$q_{k+1} \leq 10000 q_k^2$



# Renormalization steps

Multi-scale, a perturbative recipe:

- Find suitable scale sequence $L_k$
- Get a "paving" structure on $G$
- Define bad events $A_k$
- Show $A_k$ "cascades"
- Chose parameter to start induction

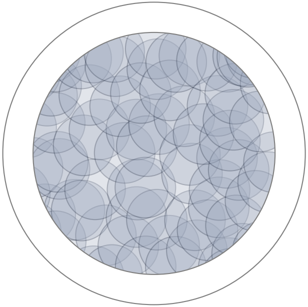
A few advantages:

- Very resilient to changes in the model
- Provides quantitative estimates

# Hypotheses

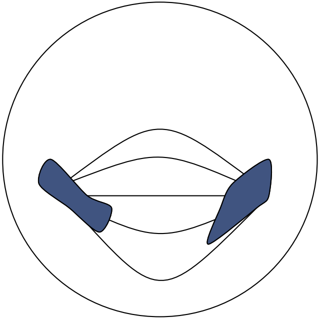
**Polynomial growth**

Allows paving:

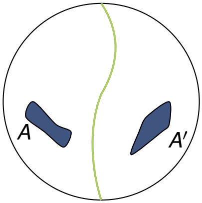


**Local isoperimetric inequality**

Big flow between sets:



# Separation events



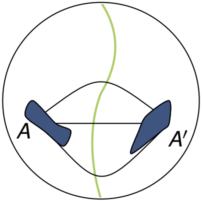

## Lemma

The separation events

$$S(x, L) = \begin{array}{c}\text{“there exist sets } A, A' \text{ in } B(x, L) \\ \text{with diameters } L/100 \\ \text{and which are not connected”}\end{array}$$

are “cascading”.

# Separation events

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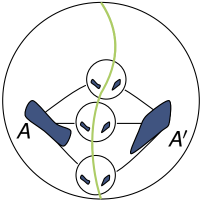


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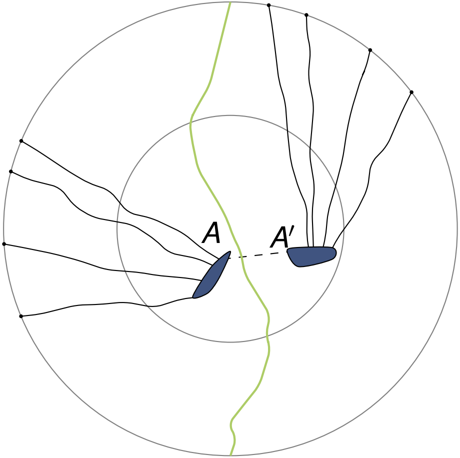
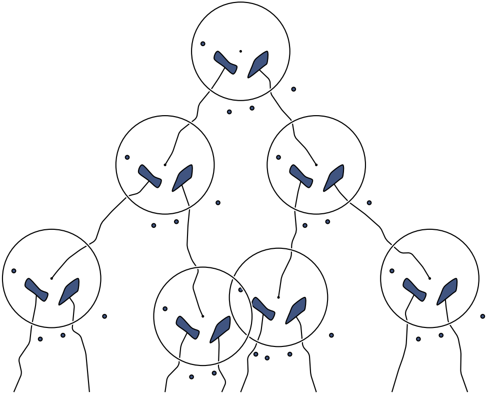
# Classic isoperimetric inequality

The "arms" from $A$ and $A'$ need not meet:



# Embedding a tree

Transitivity implies a tree could be embedded in $G$:



Contradicting polynomial growth.

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Thank you!