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Universal versus Non universal features in random matrix theory via deformed ensembles.

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Joint work with M. Capitaine; A. Guionnet and A. Edelman

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Plan

- I. Known universality results and the questions.
- II. Models and results.
- III. Some ideas of the proof.

Gaussian normality in Random Matrix Theory

The seminal result of Wigner : a Wigner real symmetric or complex Hermitian random matrix $H = H^*$ of size $n \times n$ with independent entries $H_{ij}, 1 \le i \le j \le n$ such that :

$$\mathbb{E} H_{ij} = 0; \quad \mathbb{E}|H_{ij}|^2 = 1; \quad \forall i, j.$$

Denote by $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ the ordered eigenvalues of $H_n = \frac{1}{\sqrt{n}} H$.
Theorem Wigner (58) For any interval I independent of n, denote

$$N_I := \sharp\{\lambda_i \in I, i = 1, \ldots, n\}.$$

Then as $n \to \infty$,

$$\frac{1}{n} N_I \overset{\text{a.s.}}{\to} \int_I \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{|x| \le 2} dx.$$

Local semi-circle law

Assume now that there exist constants $C, c > 0$ such that for all $1 \leq i \leq j \leq n$,

$$\mathbb{P}\left(|H_{ij}| \geq x\right) \leq Ce^{-x^c}.$$

Theorem Tao-Vu ('10) Erdòs-Knowles-Yau ('11), Erdòs-Yin-Yau ('12)
Let $\epsilon > 0$ be given. Assume that $|I| \geq n^{-1+\epsilon}$. Then one has

$$\left| N_I - n \int_I \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{|x| \leq 2} dx \right| = o(n|I|),$$

with probability greater than $1 - n^{-A}$ for any $A > 0$.

Set $\gamma_i := \inf\{y, \int_{]-\infty, y]} \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{|x| \leq 2} dx = i/n\}$, and fix $\eta > 0$ one has that

$$\lambda_i = \gamma_i + O(n^{-1+\epsilon}), \forall \eta n \leq i \leq (1 - \eta) n$$

with high probability.

Rigidity and universality of local eigenvalue statistics

The limiting distribution of the smallest, largest eigenvalues and spacings between nearest neighbor eigenvalues in the bulk of the spectrum is universal in the large n-limit and the same as for a matrix with Gaussian entries.

Theorem Tao-Vu ('09), Tao-Vu-Erdòs-Schlein-Yau ('09), Erdòs-Yin-Yau ('12) Let $f \in L^\infty(\mathbb{R}^m)$ be a symmetric compactly supported function and $u \in [-2+\epsilon', 2-\epsilon']$. Assume H is complex Hermitian. Set

$$S_n^m(f,u) = \sum_{i_1,\ldots,i_m} f(\rho_n(\lambda_{i_1}-u),\ldots,\rho_n(\lambda_{i_m}-u)),$$

where the sum bears on distinct indices in $\{1,\ldots,n\}$ and $\rho_n = n\frac{1}{2\pi}\sqrt{4-u^2}$. Then

$$\lim_{n\to\infty} \frac{1}{2\epsilon}\int_{u-\epsilon}^{u+\epsilon} \mathbb{E}S_n^m(f,u)du = \int_{\mathbb{R}^m} f(t_1,\ldots,t_m)\det\left(K_{Sin}(t_i,t_j)\right)_{i,j=1}^m dt_1\cdots dt_m,$$

where $K_{Sin}(x_i,x_j) = \dfrac{\sin\pi(x_i-x_j)}{\pi(x_i-x_j)}$ Sine Kernel.

Tools for universality

Start from a reference ensemble for which explicit computations can be done, typically Gaussian ensembles. Then either :

- Add a small (n dependent) Gaussian matrix: Gaussian divisible ensembles. Easier in the complex case where all eigenvalue statistics can be computed. Idea developed by Erdòs-Schlein-Yau ('09-).

- Use the four moment theorem : compare eigenvalue statistics to those well-known of a Gaussian matrix by assuming the moments of the entries match up to order 4. Idea developed by Tao-Vu ('09).

Or mix the two methods.

The questions

- The universality results are asymptotic: what can be said for finite n?
  There is a need to quantify the effect of finite n.

- What (and where) is the impact of the non Gaussianity on eigenvalue statistics? To study this question one needs a reference ensemble which is not Gaussian.

- Need for more general ensembles than Gaussian ensembles.

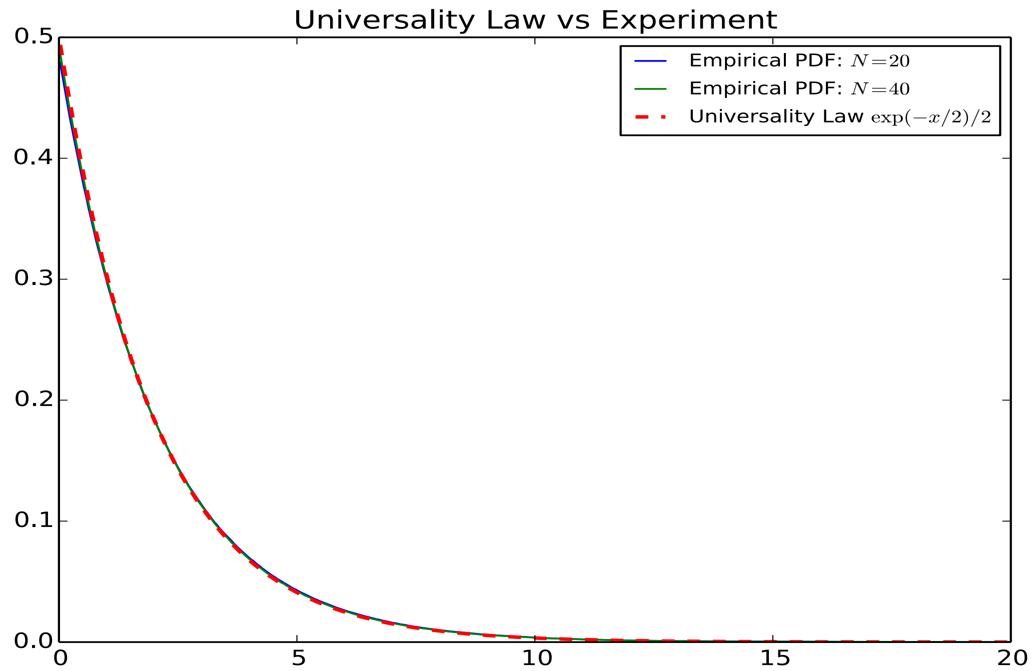
Consider both Wigner $H/\sqrt{n}$ and sample covariance matrices MM^*/n.

Sample covariance matrix : M of size $n \times p$, $p - n = \nu$ fixed. M has independent real or complex entries

$$\mathbb{E}M_{ij} = 0; \quad \mathbb{E}|M_{ij}|^2 = \sigma^2, \quad \forall 1 \leq i \leq n, 1 \leq j \leq p. \text{ Then form } MM^*/n.$$

The limiting e.e.d. is then the Marcenko-Pastur distribution.

Finite n distribution of the smallest eigenvalue of a sample covariance matrix



Universality Law vs Experiment: $n = 20$ and $n = 40$ already resemble $n = \infty$

A first non universal asymptotic result

Assume in addition that :
-all the entries share the same fourth moment $M = \mathbb{E}|H_{ij}|^4$.
-For any $\eta > 0$,

$$\frac{1}{\eta^4 n^2} \sum_{i,j} \mathbb{E}|H_{ij}|^4 \mathbb{1}_{|H_{ij}| \geq \eta\sqrt{n}} = o(1).$$

Theorem: Bai-Yao ('05)
Let f be an analytic function $f : \mathcal{U} \to \mathbb{C}$ where $\mathcal{U}$ is an open complex set containing $[-2, 2]$. Then

$$\sum f(\lambda_i) - n \int f(x) \frac{1}{2\pi} \sqrt{4 - x^2} dx \to G,$$

whose mean value is explicit and vanishes only in the Gaussian case (variance also known).

A hint for non universality of individual eigenvalues

Tao-Vu ['11]: Compute

$$\mathbb{E}\sum_{i=1}^{n}\lambda_i^4 = \mathbb{E}\mathrm{Tr}H_n^4 = \frac{2(n-1)(n-2)}{n}(1+o(1))\sigma^4 + \frac{n-1}{n}\mathbb{E}\Big|H_{12}\Big|^4 + o(1).$$

Theorem: Tao-Vu ('11) For two matrices H, H' whose entries admit different moments of order 4 (and vanishing third moment), for n large enough,

$$\sum_{i=1}^{n}|\mathbb{E}\lambda_i - \mathbb{E}\lambda_i'| \geq \kappa$$

for some $\kappa > 0$.

On average the mean value of eigenvalues has to depend on the fourth moment.

Model: complex Gaussian divisible ensembles

$$M = W + aV \begin{cases} M = M^* \text{ and size } n \times n \text{ if Wigner} \\ M \text{ of size } n \times p; p - n = \nu \text{ fixed integer if sample covariance} \end{cases} \quad s.t.$$

- V is a random matrix with i.i.d. (modulo the symmetry) complex $\mathcal{N}(0,1)$ entries;
- $a > 0$ is some real parameter
- W is a complex deterministic or random matrix. If random the entries have law P_{ij}: P_{ij} has uniform sub-exponential decay and

$$\mathbb{E}W_{ij} = 0, \quad \mathbb{E}|W_{ij}|^2 = 1/4 \text{ and } M := \mathbb{E}|W_{ij}|^4, \forall i, j.$$

We consider the asymptotic $1/n$ expansion of local eigenvalue statistics:
- at the hard edge: M of size $n \times p$, $p - n$ fixed and consider $X = MM^*/n$
- in the bulk of the spectrum: M of size $n \times n$ and set $X = M/\sqrt{n}$.

The macroscopic effect of finite n at the soft edges

Consider a deterministic matrix W whose empirical eigenvalue distribution converges to some probability ν as $n \to \infty$. Form the matrix

$$M := W + \sigma V n^{-1/2}, \text{ where } V \text{ is non necessarily Gaussian Wigner r.m..}$$

Theorem : Pastur-Vasilchuk ('00), Biane (1997); Sample covariance: Marcenko-Pastur (1967), Silverstein (1995).
The empirical eigenvalue distribution of M converges weakly as $n \to \infty$ to a probability measure μ.
μ is defined by its Stieltjes transform S_μ : $\quad z \in \mathbb{C} \setminus \mathbb{R} \mapsto \int \frac{1}{z-y} d\mu(y)$:

$$S_\mu(z) = S_\nu \left(\frac{1}{z - \sigma^2 S_\mu(z)} \right).$$

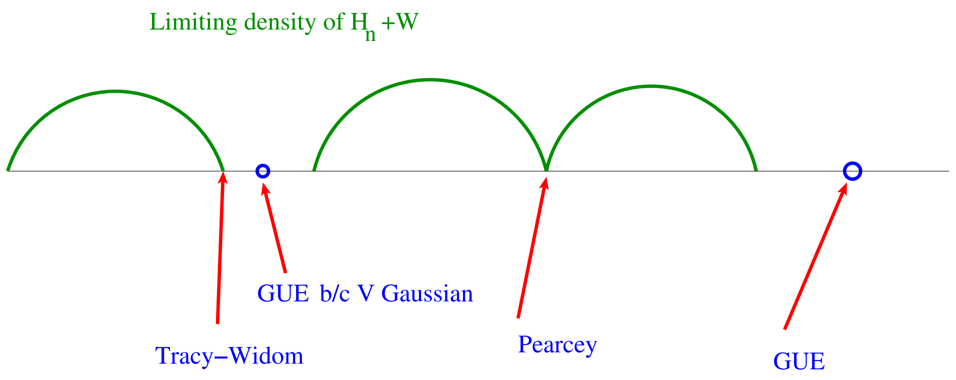
μ is called the free convolution of ν and the semi-circle distribution. The support may have disjoint connected components.

Asymptotic behavior of extreme eigenvalues

Theorem : Capitaine-Donati Martin-Féral-Février ('11). Denote by $\theta_i, i = 1, \ldots, r$ the eigenvalues (independent of n) of W outside supp(ν). Assume $\sup_{i,j} \mathbb{E}|H_{ij}|^4 < \infty$.

$$\forall \theta_i \text{ s.t. } \int \frac{1}{(\theta_i - y)^2} d\nu(y) < \frac{1}{\sigma^2}, \text{ corresponds } \lambda_i \notin \text{ supp}(\mu).$$

Theorem : T. Scherbina ('13), Capitaine-P ('14), Capitaine Donati-Martin Féral ('09); SC : Baik-Rao ('14), Hachem-Hardy-Najim ('14), Lee-Schnelli ('14) random, Knowles-Yin ('15)
Let V be GUE and assume that the non spiked eigenvalues of W denoted by $y_j(W)$, $j = 1, \ldots, N - r$, satisfy $\max_{1 \le j \le N-r} \mathrm{dist}(y_j(W), \mathrm{supp}(\nu)) \to_{n \to \infty} 0$.



A restriction for Tracy-Widom :
$\forall t \in$ supp (ν),
$\lim_{\epsilon \to 0} \int \frac{d\nu(x)}{(t + i\epsilon - x)^2} > \sigma^{-2}$.

Other parts of the spectrum: W random, V Gaussian

Theorem Edelman-Guionnet-P ('14): the hard edge of sample covariance matrices
Let F_n be the cumulative density function at the hard edge in the Gaussian case with entries with complex variance $\sigma^2 = 1/4 + a^2$:

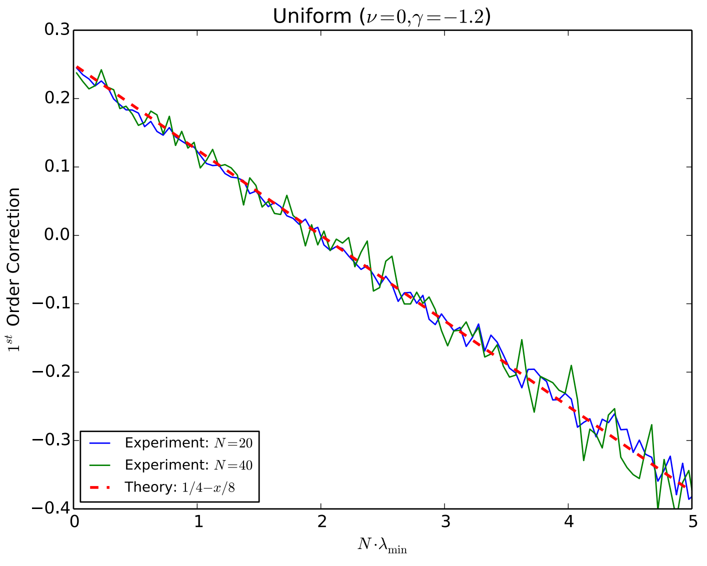
$$F_n(s) = \mathbb{P}\left(\sigma^2 \lambda_{min}(VV^*) \leq \frac{s}{n}\right)$$

Then, for all $s > 0$, if P_{ij}'s have complex fourth cumulant $\kappa_4 = \int |zz^*| dP_{ij}(z) - 8\sigma_{\mathbb{R}}^2$

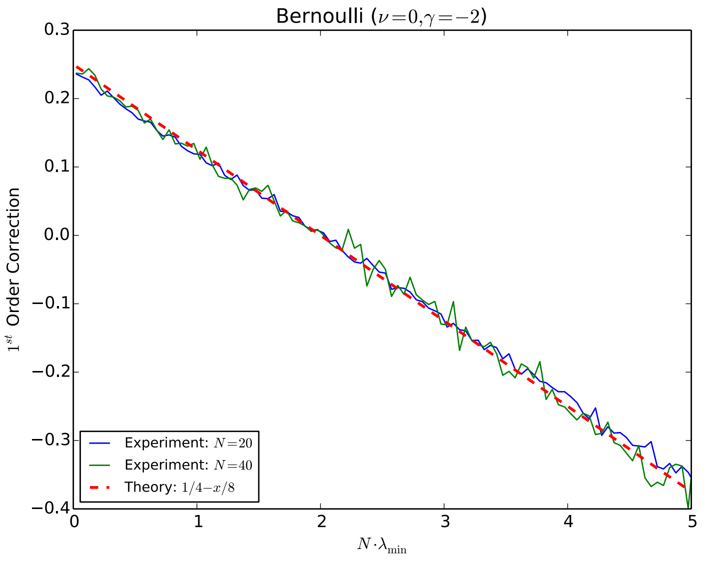
$$\mathbb{P}\left(\lambda_{min}(MM^*) \leq \frac{s}{n}\right) = F_n(s) + \frac{sF_n'(s)}{\sigma^4 n}\kappa_4 + o\left(\frac{1}{n}\right).$$

Corollary (Schehr ('14), Bornemann ('15)) Let $F_\infty(s) = \lim_{n\to\infty} F_n(s)$.

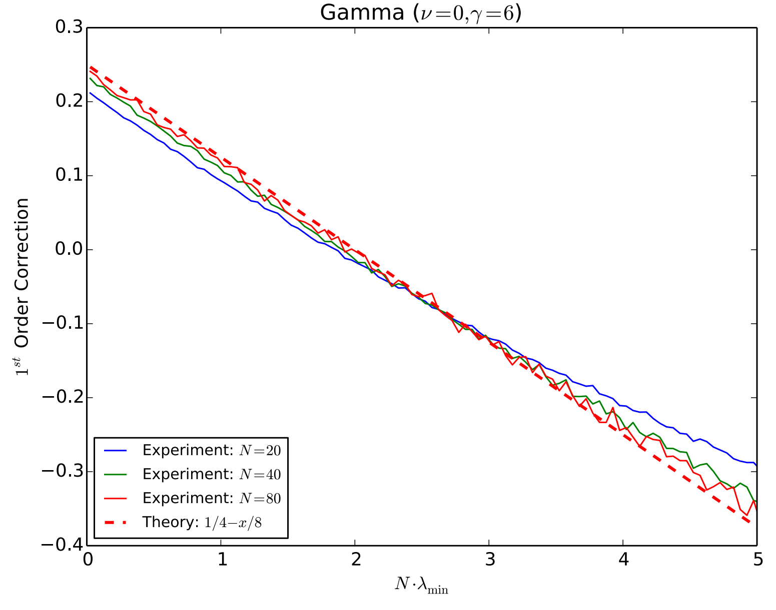
$$\mathbb{P}\left(\lambda_{min}(MM^*) \leq \frac{s}{n}\right) = F_\infty(s) + \left(\nu + \frac{\kappa_4}{\sigma^4}\right)\frac{sF_\infty'(s)}{n} + o\left(\frac{1}{n}\right).$$



Uniform ($\nu=0,\gamma=-1.2$)
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
0
1
2
3
4
5
1^{st} Order Correction
$N\cdot\lambda_{\min}$
Experiment: $N=20$
Experiment: $N=40$
Theory: $1/4-x/8$



Bernoulli ($\nu=0,\gamma=-2$)
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
0
1
2
3
4
5
1^{st} Order Correction
$N\cdot\lambda_{\min}$
Experiment: $N=20$
Experiment: $N=40$
Theory: $1/4-x/8$



Gamma ($\nu=0,\gamma=6$)
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
0
1
2
3
4
5
1^{st} Order Correction
$N\cdot\lambda_{\min}$
Experiment: $N=20$
Experiment: $N=40$
Experiment: $N=80$
Theory: $1/4-x/8$

Some results in the bulk

Let ρ_n be the one point correlation function induced by the Wigner Gaussian divisible ensemble.

Recall $\forall f$ compactly supported $\mathbb{E}\frac{1}{n}\sum_{i=1}^n f(\lambda_i) = \int_{\mathbb{R}} f(x)\rho_n(x)dx$.

Theorem E-G-P ('14) Let $\varepsilon > 0$. There exist universal functions C, D such that

$$\forall\, x \in [-2\sigma+\varepsilon, 2\sigma-\varepsilon],\ \rho_n(x) = \sigma_{sc}(x) + \frac{1}{n}C(x) + \frac{1}{n}\kappa_4 D(x) + o\left(\frac{1}{n}\right).$$

Define $N_n(x) := \frac{1}{n}\sharp\{i, \lambda_i \leq x\}$, and $\hat{\gamma}_i$ by

$$\hat{\gamma}_i := \inf\left\{y, \mathbb{E}N_n(y) = \frac{i}{n}\right\}.$$

Then for all $i \in [n\varepsilon, n(1-\varepsilon)]$ for some $\varepsilon > 0$, there exists a universal constant C_i so that

$$\hat{\gamma}_i - \gamma_i = \frac{C_i}{n} + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + o(\frac{1}{n}).$$

Ideas of the proof

Fix W and set $H := WW^*/n$. Let $y_i(H), i = 1, \dots n$ denote its eigenvalues.

Theorem Guhr-Wettig, Jackson-Sener-Verbaarschot (96) The induced joint eigenvalue distribution has a density w.r.t. Lebesgue measure given by

$$g(x_1, \dots, x_n; y(H)) = \frac{\Delta(x)}{\Delta(y(H))} \det\left(\frac{e^{-\frac{y_i(H)+x_j}{2t}}}{2t} I_\nu \left(\frac{\sqrt{y_i(H)x_j}}{t} \right) \left(\frac{x_j}{y_i(H)} \right)^{\frac{\nu}{2}} \right)_{i,j=1}^n,$$

where $t = \frac{a^2}{2n}$, and $\Delta(x) = \prod_{i<j}(x_i - x_j)$. The density induces a determinantal RPF:

$$\begin{aligned} R_m(u_1, \dots, u_m; y(H)) &= \frac{n!}{(n-m)!} \int_{\mathbb{R}_+^{n-m}} g(u_1, \dots, u_n; y(H)) \prod_{i=m+1}^n du_i \\ &= (\det K_n(u_i, u_j; y(H)))_{i,j=1}^m . \end{aligned}$$

Ideas of the proof II

The correlation kernel is

$$K_n(u,v;y(H)) = \frac{e^{\nu i\pi}}{i\pi s^3}\int_\Gamma\int_\gamma dwdz\; w\; z\; K_B\left(\frac{2znu^{1/2}}{a^2},\frac{2wnv^{1/2}}{a^2}\right)\left(\frac{w}{z}\right)^\nu \tilde{g}(w,z)e^{\{nG_n(w)-nG_n(z)\}}$$

where $K_B(x,y)=\frac{xI_\nu^{'}(x)I_\nu(y)-yI_\nu^{'}(y)I_\nu(x)}{x^2-y^2}$, $\tilde{g}(w,z)=\frac{a^2}{2}\frac{wG_n'(w)-zG_n'(z)}{w^2-z^2}$, and

$$G_n(w)=w^2/a^2+\frac{1}{n}\sum_{i=1}^{n}\ln(w^2-y_i(H));$$

In particular one has

$$\mathbb{P}\left(\lambda_{min}\geq\frac{\sigma^2 s}{4n^2}\right)=\int d\mathbb{P}_n(H)\det(I-K_n(\cdot;y(H)))_{L^2(0,\frac{\sigma^2 s}{4n^2})}.$$

Ideas of the proof III

At the hard edge $u, v \sim 1/n^2$: the leading exponential term is given by G_n. Because $1/n \sum \delta_{y_i(H)} \to \rho_{MP}$, one has that with high probability and for $w \in \mathbb{C} \setminus \mathbb{R}$,

$$G_n \sim G(w) := w^2/a^2 + \int \ln(w^2 - y) d\rho_{MP}(y).$$

Introduce the two true (resp. approximate) critical points: $z_c^{\pm}, (w_c^{\pm}) \in i\mathbb{R}$ s.t.

$$G_n'(z_c^{\pm}) = 0; \quad G'(w_c^{\pm}) = 0. \text{ Then } |z_c^{\pm} - w_c^{\pm}| = o(n^{-c}).$$

First order asymptotics by a saddle point: universality (Ben Arous -Peche ('05); (Forrester (93) for LUE):

$$\lim_{n\to\infty} \mathbb{P}\left(\lambda_{min} \geq \frac{\sigma^2 s}{4n^2}\right) \to \det(I - \widetilde{K_B})_{L^2(0,s)},$$

where $\widetilde{K_B}$ is the usual Bessel kernel $\widetilde{K_B}(u,v) := e^{\nu i\pi} K_B(i\sqrt{u}, i\sqrt{v})$.

Ideas of the proof IV

Second order asymptotics up to order $1/n$:
There exists a **smooth and universal** function A such that for all u, v

$$\frac{\sigma^2}{4n^2} K_n\left(\frac{u\sigma^2}{4n^2}, \frac{v\sigma^2}{4n^2}; y(H)\right) = \widetilde{K_B}(u,v) + \frac{A(u,v)}{n} + \left(\left(\frac{z_c^+}{w_c^+}\right)^2 - 1\right)\frac{\partial}{\partial\beta}\Big|_{\beta=1} \beta\widetilde{K_B}(\beta u, \beta v) + o\left(\frac{1}{n}\right).$$

Given $z \in \mathbb{C}\backslash\mathbb{R}_+$, set

$$X_n(z) = \sum_{i=1}^{n} \frac{1}{y_i(H) - z} - nS_{\mathrm{MP}}(z).$$

Then,

$$\left(\frac{z_c^+}{w_c^+}\right)^2 - 1 = -\frac{1}{(w_c^+)^2 S'_{\mathrm{MP}}((w_c^+)^2)} \frac{1}{n} X_n((w_c^+)^2) + o\left(\frac{1}{n}\right).$$

Ideas of the proof V

We can then deduce that there exists a non-negative function g_n^0, depending on n but not on the P_{jk}'s, so that

$$\mathbb{P}\left(\lambda_{min} \geq \frac{\sigma^2 s}{4n^2}\right) = g_n^0(s) + \frac{1}{n}\partial_\beta g_n^0(\beta s)|_{\beta=1} \int dP_n(H)[\Delta_n(H)] + o\left(\frac{1}{n}\right)$$

where

$$\Delta_n(H) = \frac{-1}{(w_c^+)^2 S'_{\mathrm{MP}}((w_c^+)^2)} X_n((w_c^+)^2).$$

The role of the fourth moment via the TCL for Stieltjes transforms:
Theorem Najim Yao ('14), Bai ('10) Let $z \in \mathbb{C} \setminus \mathbb{R}_+ \cap \{Z \in \mathbb{C}, \Im Z \geq 0\}$. One has that

$$\lim_{n\to\infty} \mathbb{E}[X_n(z)] = A(z) - \kappa_4 B(z),$$

with A independent of κ_4, and $B(z) = \frac{S_{\mathrm{MP}}(z)^2}{(1+\frac{S_{\mathrm{MP}}(z)}{4})^2(z+\frac{zS_{\mathrm{MP}}(z)}{2})}$.

Conclusion

- Gaussian divisible ensembles are the simplest ensembles for which both explicit computations and non Gaussianity can be considered: a reference non Gaussian ensemble. However one restricts to complex ensembles.

- The soft edge has not been studied: due to possible outliers, one would need strong concentration results. In principle this can be done with minor modifications.

- We have proved a (weaker) version of Tao-Vu's conjecture stating that

$$\mathbb{E}\lambda_i - \gamma_i = \frac{C_i}{n} + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + o(\frac{1}{n}).$$

- No a priori result for real ensembles.