OXFORD SPA 2015

Haya Kaspi

(Technion, Israel Institute of Technology)

38th Conference on Stochastic Processes and their Applications

Spa2015@oxford-man.ox.ac.uk

Measure Valued Processes and Skorohod Maps that act on them in the Analysis and Control of Non Markovian Queues

Rami Atar, Anup Biswas, Haya Kaspi and Kavita Ramanan

July 9, 2015

Stochastic Processes and their Applications Oxford 2015

Lecture Outline

In the past decade measure valued processes have become an efficient tool in studying limits of queueing systems.

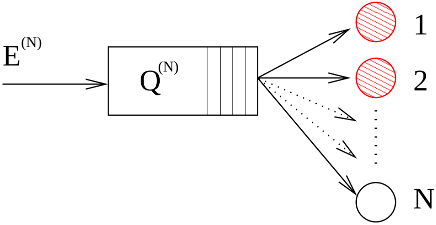
In this lecture we shall treat two quite different examples:

- The many Servers Queue $G/G/N$ as $N \to \infty$.
- The Single Server Earlier Deadline First (EDF) queue in heavy traffic.

Analysis of Queueing Systems Using Measure Valued Processes

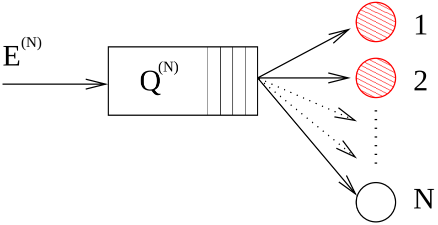
- The single server EDF queue without reneging B. Doytchinow, J. Lehotczky and S. Shreve 2001,
- Processor sharing queues with non exponential service times–C.Gromoll, A. Puha and R. Williams 2002
- Single server EDF L. Decreusefond and P. Moyal,2008
- $G/G/\infty$ L. Decreusefond and P. Moyal 2009
- The single server EDF queue with reneging L. Kruk, J. Lehotzky, S. Shreve and K.Ramanan 2011.
- Many Servers Queue (i)H. Kaspi and K. Ramanan 2010, (ii) Kang and Ramanan 2010(iii)J. Zhang 2012.

The Many Servers Queue



- Cumulative arrival process $E^{(N)}$ a counting process independent of the service times of customers and of the number of customers in the system at time 0.
- Service times of various customers are i.i.d variables with distribution G having density g mean 1 and hazard rate function $h = \frac{g}{1-G}$.
- N servers.
- First Come First Serve (FCFS) Non Idling.
- $X^{(N)}(t)$ the number of customers in the system at time t.
- $I^{(N)}(t)$ the number of idle servers at time t.

The Many Servers Queue



- Cumulative arrival process $E^{(N)}$ a counting process independent of the service times of customers and of the number of customers in the system at time 0.
- Service times of various customers are i.i.d variables with distribution G having density g mean 1 and hazard rate function $h = \frac{g}{1-G}$.
- N servers.
- First Come First Serve (FCFS) Non Idling.
- $X^{(N)}(t)$ the number of customers in the system at time t.
- $I^{(N)}(t)$ the number of idle servers at time t.

Some Relevant Prior and Related Work

- Halfin and Whitt 1981 ($\lambda_N = N - \beta\sqrt{N}$ for some $\beta > 0$)
- Puhalskii Reimann 2000
- Reed 2007, Reed 2008, Puhalskii and Reed 2009
- Jelenkovic, Mandelbaum and Momcilovic 2002
- Gamarnik and Momcilovic 2008

Markovian State Descriptor of the N-System

$$(R_E^{(N)}(t), X^{(N)}(t), \nu^{(N)}(t))$$

- $R_E^{(N)}(t)$ the time from t until the next arrival into the system–assumed to be a Markov process with respect its own filtration.
- $X^{(N)}(t)$ the number of customers in the system–in service and in queue.
- $\nu_t^{(N)}$ a counting measure on $\mathbb{R}_+$ that puts point masses at the ages (times in service) of the various customers that are in service at time t.
- Let $(\mathcal{F}_t^{(N)})$ be the filtration generated by those three processes.

Some additional processes:

- $a_j^{(N)}(t)$ the age, at time t, of the j-th customer to enter the service. We use negative j's for customers that were in service at time 0.
- $\langle 1, \nu_t^{(N)} \rangle$ the number of customers in service at time t.
- $K_t^{(N)}$ the number of customers that have entered the service in $(0, t]$.

The Measure $\nu^{(N)}$

Then

$$\nu_t^{(N)} = \sum_{j=-\langle 1,\nu_0^{(N)}\rangle+1}^{K^{(N)}(t)} \delta_{a_j^{(N)}(t)} \mathbb{1}_{\{a_j^{(N)}(t)<v_j\}}$$

where v_j is the (original) service requirement of the j-th customer and δ_a is the Dirac mass at the point a.

The number of idle servers at time t is:

$$I^{(N)}(t) = N - \langle 1, \nu_t^{(N)} \rangle$$

and the non idling condition reads

$$N - \langle 1, \nu_t^{(N)} \rangle = [N - X^{(N)}(t)]^+.$$

The Measure $\nu^{(N)}$

Then

$$\nu_t^{(N)} = \sum_{j=-\langle 1,\nu_0^{(N)}\rangle+1}^{K^{(N)}(t)} \delta_{a_j^{(N)}(t)} \mathbb{1}_{\{a_j^{(N)}(t)<v_j\}}$$

where v_j is the (original) service requirement of the j-th customer and δ_a is the Dirac mass at the point a.

The number of idle servers at time t is:

$$I^{(N)}(t) = N - \langle 1, \nu_t^{(N)} \rangle$$

and the non idling condition reads

$$N - \langle 1, \nu_t^{(N)} \rangle = [N - X^{(N)}(t)]^+.$$

The Departure process

Let $D^{(N)}(t)$ be the cumulative departure up to time t. $D^{(N)}(t)$ is a point process. Mark it with a function φ in $\mathcal{C}_b^1([0,L)\times\mathbb{R}_+)$. Define,

- $$D_\varphi^{(N)}(t) = \sum_{s\in[0,t]} \sum_{j=-\langle 1,\nu_0^{(N)}\rangle+1}^{K^{(N)}(t)} \mathbb{1}_{\{\frac{da_j^{(N)}}{ds}(s-)>0, a_j^{(N)}(s)=v_j\}} \varphi\left(a_j^{(N)}(s), s\right).$$

- For $\varphi = \mathbb{1}_{[0,x]}$, $D_\varphi^{(N)}$ = number of departing customers in $[0,t]$ with age $\leq x$. Note that this is also the number of departing customers in $[0,t]$ whose service requirements were smaller than x.

- For $\varphi = \mathbf{1}$, $D_{\mathbf{1}}^{(N)}(t) = D^{(N)}(t)$.

The Departure process

Let $D^{(N)}(t)$ be the cumulative departure up to time t. $D^{(N)}(t)$ is a point process. Mark it with a function φ in $\mathcal{C}_b^1([0,L) \times \mathbb{R}_+)$. Define,

- $$D_\varphi^{(N)}(t) = \sum_{s \in [0,t]} \sum_{j=-\langle 1, \nu_0^{(N)} \rangle + 1}^{K^{(N)}(t)} \mathbb{1}_{\left\{ \frac{da_j^{(N)}}{ds}(s-) > 0, a_j^{(N)}(s) = v_j \right\}} \varphi\left(a_j^{(N)}(s), s \right).$$

- For $\varphi = \mathbb{1}_{[0,x]}$, $D_\varphi^{(N)}$ = number of departing customers in $[0,t]$ with age $\leq x$. Note that this is also the number of departing customers in $[0,t]$ whose service requirements were smaller than x.

- For $\varphi = \mathbf{1}$, $D_{\mathbf{1}}^{(N)}(t) = D^{(N)}(t)$.

The compensator

The departure process $D_{\mathbf{1}}^{(N)}$ is a sum of jumps, its $(\mathcal{F}_t)$ compensator is given by

$$A^{(N)}(t) = \int_0^t \langle h, \overline{\nu}_s^{(N)} \rangle ds$$

where

$$\langle h, \nu_s^{(N)} \rangle = \int_{\mathbb{R}_+} h(x) \nu_s^{(N)}(dx),$$

Similarly, the compensator of $D_{\varphi}^{(N)}$ is equal to

$$A_{\varphi}^{(N)}(t) = \int_0^t \langle \varphi h, \overline{\nu}_s^{(N)} \rangle ds$$

where we recall that h is the hazard rate function of the service distribution

$$h = \frac{g}{1 - G}$$

The Process Dynamics

For $\varphi \in \mathcal{C}_b^1([0,L) \times \mathbb{R}_+)$, $t \geq 0$, define

$$\langle \varphi(\cdot,t), \nu_t^{(N)} \rangle = \int_{[0,\infty)} \varphi(x,t)\nu_t^{(N)}(dx)$$

- $$\langle \varphi(\cdot,t), \nu_t^{(N)} \rangle = \langle \varphi(\cdot,t), \nu_0^{(N)} \rangle + \int_0^t \langle \varphi_x(\cdot,s) + \varphi_t(\cdot,s), \nu_s^{(N)} \rangle ds$$
$$-(D_\varphi^{(N)}(t) - A_\varphi^{(N)}(t)) + A_\varphi^{(N)}(t) + \int_0^t \varphi(0,u)dK^{(N)}(u)$$

- $$X^{(N)}(t) = X^{(N)}(0) + E^{(N)}(t) - (D_1^{(N)}(t) - A_1^{(N)}(t)) + A_1^{(N)}(t)$$

- $$N - \langle 1, \nu_t^{(N)} \rangle = \left(N - X^{(N)}(t)\right)^+$$

The Process Dynamics

For $\varphi \in \mathcal{C}_b^1([0,L) \times \mathbb{R}_+)$, $t \geq 0$, define

$$\langle \varphi(\cdot,t), \nu_t^{(N)} \rangle = \int_{[0,\infty)} \varphi(x,t)\nu_t^{(N)}(dx)$$

- $$\langle \varphi(\cdot,t), \nu_t^{(N)} \rangle = \langle \varphi(\cdot,t), \nu_0^{(N)} \rangle + \int_0^t \langle \varphi_x(\cdot,s) + \varphi_t(\cdot,s), \nu_s^{(N)} \rangle ds$$
$$-(D_\varphi^{(N)}(t) - A_\varphi^{(N)}(t)) + A_\varphi^{(N)}(t) + \int_0^t \varphi(0,u)dK^{(N)}(u)$$

- $$X^{(N)}(t) = X^{(N)}(0) + E^{(N)}(t) - (D_1^{(N)}(t) - A_1^{(N)}(t)) + A_1^{(N)}(t)$$

- $$N - \langle 1, \nu_t^{(N)} \rangle = \left(N - X^{(N)}(t)\right)^+$$

Fluid Scaling

We scale by dividing $X^{(N)}, \nu^{(N)}, E^{(N)}, K^{(N)}, D^{(N)}, I^{(N)}$ by N.
Let $\overline{X}^{(N)}, \overline{\nu}^{(N)}, \overline{E}^{(N)}, \overline{K}^{(N)}, \overline{D}^{(N)}, \overline{I}^{(N)}$ be the resulting processes.

Assumptions

- $$\overline{E}^{(N)} \to \overline{E} \quad \text{in} \quad D[0,\infty) \quad \text{and} \quad E(\overline{E}(t)) < \infty \text{ for all } \quad t \geq 0$$
- $$\overline{X}^{(N)}(0) \to \overline{X}(0) \text{ a.s. and } \quad E(\overline{X}(0)) < \infty.$$
- $$\overline{\nu}_0^{(N)} \to \overline{\nu}_0 \quad \text{in the space of finite measures on} \quad [0, L)$$

where

$$L = \inf\{t : G(t) = 1\}.$$

Fluid Scaling

We scale by dividing $X^{(N)}, \nu^{(N)}, E^{(N)}, K^{(N)}, D^{(N)}, I^{(N)}$ by N.
Let $\overline{X}^{(N)}, \overline{\nu}^{(N)}, \overline{E}^{(N)}, \overline{K}^{(N)}, \overline{D}^{(N)}, \overline{I}^{(N)}$ be the resulting processes.

Assumptions

- $$\overline{E}^{(N)} \to \overline{E} \quad \text{in} \quad D[0,\infty) \quad \text{and} \quad E(\overline{E}(t)) < \infty \text{ for all } \quad t \geq 0$$
- $$\overline{X}^{(N)}(0) \to \overline{X}(0) \text{ a.s. and } \quad E(\overline{X}(0)) < \infty.$$
- $$\overline{\nu}_0^{(N)} \to \overline{\nu}_0 \quad \text{in the space of finite measures on} \quad [0, L)$$

where

$$L = \inf\{t : G(t) = 1\}.$$

The Fluid Equation

A rcll function $(\overline{X}, \overline{\nu})$ on $[0, \infty)$ with first component with values in $\mathbb{R}_+$ and second in the space of sub-probability measures, is said to solve the fluid equation associated with $(\overline{E}, \overline{X}(0), \overline{\nu}_0)$ and h, if for every $t \geq 0$,

- $\int_0^t \langle h, \overline{\nu}_s \rangle ds < \infty$ and the following are satisfied for every test function $\varphi \in \mathcal{C}_b^1([0, L) \times \mathbb{R}_+)$

-

$$\langle \varphi(\cdot, t), \overline{\nu}_t \rangle = \langle \varphi(\cdot, t), \overline{\nu}_0 \rangle + \int_0^t \langle \varphi_x(\cdot, s) + \varphi_t(\cdot, s), \overline{\nu}_s \rangle ds - \int_0^t \langle h(\cdot)\varphi(\cdot, s), \overline{\nu}_s \rangle ds + \int_0^t \varphi(0, u) d\overline{K}(u).$$

The Fluid Equation

A rcll function $(\overline{X}, \overline{\nu})$ on $[0, \infty)$ with first component with values in $\mathbb{R}_+$ and second in the space of sub-probability measures, is said to solve the fluid equation associated with $(\overline{E}, \overline{X}(0), \overline{\nu}_0)$ and h, if for every $t \geq 0$,

- $\int_0^t \langle h, \overline{\nu}_s \rangle ds < \infty$ and the following are satisfied for every test function $\varphi \in \mathcal{C}_b^1([0, L) \times \mathbb{R}_+)$
-

$$\langle \varphi(\cdot, t), \overline{\nu}_t \rangle = \langle \varphi(\cdot, t), \overline{\nu}_0 \rangle + \int_0^t \langle \varphi_x(\cdot, s) + \varphi_t(\cdot, s), \overline{\nu}_s \rangle ds - \int_0^t \langle h(\cdot)\varphi(\cdot, s), \overline{\nu}_s \rangle ds + \int_0^t \varphi(0, u) d\overline{K}(u).$$

- $$\overline{X}(t) = \overline{X}(0) + \overline{E}(t) - \int_0^t \langle h, \overline{\nu}_s \rangle ds$$

- $$1 - \langle 1, \overline{\nu}_t \rangle = (1 - \overline{X}(t))^+$$

- $$\overline{K}(t) = \langle 1, \overline{\nu}_t \rangle - \langle 1, \overline{\nu}_0 \rangle + \int_0^t \langle h, \overline{\nu}_s \rangle ds$$

- $$\overline{X}(t) = \overline{X}(0) + \overline{E}(t) - \int_0^t \langle h, \overline{\nu}_s \rangle ds$$

- $$1 - \langle 1, \overline{\nu}_t \rangle = (1 - \overline{X}(t))^+$$

- $$\overline{K}(t) = \langle 1, \overline{\nu}_t \rangle - \langle 1, \overline{\nu}_0 \rangle + \int_0^t \langle h, \overline{\nu}_s \rangle ds$$

Theorem 1. Suppose the basic assumptions are satisfied with $(\overline{E}, \overline{\nu}_0, \overline{X}(0))$ Then

- $(\overline{X}^{(N)}, \overline{\nu}^{(N)})$ converges in probability to the unique solution $(\overline{X}, \overline{\nu})$ of the fluid equation. Moreover, the solution satisfies for every $f \in \mathcal{C}_b([0,M))$

$$\begin{aligned}\langle f, \overline{\nu}_t \rangle = \int f(x)\overline{\nu}_t(dx) &= \int_{[0,M)} f(x+t)\frac{1-G(x+t)}{1-G(x)}\overline{\nu}_0(dx) \\ &+ \int_0^t f(t-s)(1-G(t-s))d\overline{K}(s)\end{aligned}$$

- If $\overline{E}$ is absolutely continuous with $\bar{\lambda}(t) = \frac{d\overline{E}(t)}{dt}$, then $\overline{K}$ is absolutely continuous and $\kappa(t) = \frac{d\overline{K}(t)}{dt}$ satisfies for a.e. $t \in [0, \infty)$,

$$\kappa(t) = \begin{cases} \bar{\lambda}(t) & \text{if} \quad \overline{X}(t) < 1 \\ \bar{\lambda}(t) \wedge \langle h, \overline{\nu}_t \rangle & \text{if} \quad \overline{X}(t) = 1 \\ \langle h, \overline{\nu}_t \rangle & \text{if} \quad \overline{X}(t) > 1. \end{cases}$$

Theorem 1. Suppose the basic assumptions are satisfied with $(\overline{E}, \overline{\nu}_0, \overline{X}(0))$ Then

- $(\overline{X}^{(N)}, \overline{\nu}^{(N)})$ converges in probability to the unique solution $(\overline{X}, \overline{\nu})$ of the fluid equation. Moreover, the solution satisfies for every $f \in \mathcal{C}_b([0, M))$

$$\begin{aligned}\langle f, \overline{\nu}_t \rangle = \int f(x)\overline{\nu}_t(dx) &= \int_{[0,M)} f(x+t)\frac{1-G(x+t)}{1-G(x)}\overline{\nu}_0(dx) \\ &+ \int_0^t f(t-s)(1-G(t-s))d\overline{K}(s)\end{aligned}$$

- If $\overline{E}$ is absolutely continuous with $\bar{\lambda}(t) = \frac{d\overline{E}(t)}{dt}$, then $\overline{K}$ is absolutely continuous and $\kappa(t) = \frac{d\overline{K}(t)}{dt}$ satisfies for a.e. $t \in [0, \infty)$,

$$\kappa(t) = \begin{cases} \bar{\lambda}(t) & \text{if} \quad \overline{X}(t) < 1 \\ \bar{\lambda}(t) \wedge \langle h, \overline{\nu}_t \rangle & \text{if} \quad \overline{X}(t) = 1 \\ \langle h, \overline{\nu}_t \rangle & \text{if} \quad \overline{X}(t) > 1. \end{cases}$$

Robustness of the Framework-Extensions

- One can add reneging from the queue by adding another measure that records the waiting times of customers in the queue. W. Kang and K. Ramanan.
- Treat multiclass queues and control them using priority rule. Optimality of an index priority control under certain assumptions. R. Atar, H.K. and N. Shimkin.
- CLT-Diffusion Limits that lead to SPDE
  H.K. and K. Ramanan

Continuous parameter parioritizing and the infinite dimensional Skorokhod Map

The Earlier Deadline First(EDF) priority single server queue

- A single server, infinite buffer.
- Arrival-a counting process. Services i.i.d. random variables with mean 1.
- Customers have deadlines on which the declare upon arrival.
- Server is non idling, works at a rate that may depend on time.
- Server prioritizes according to the earlier deadline first in a non preemptive manner.

Two Types of EDF Systems

- soft deadlines Customers do not renege when their deadlines are met, they just accumulate lateness.
- hard deadlines Customers renege when their deadlines are met before being admitted into service. Customers do not renege while being served.

The N System

We work with a sequence of Hard-EDF single server queues indexed by $N \in \mathbb{N}$.

Processes associated with the N-system:

- ξ_t^N a measure valued(on $\mathbb{R}_+$) processes in $\mathbb{D}_{\mathcal{M}}$. For $t \geq 0, x \geq 0 \quad \xi_t^N[0,x]$ is the number of customers in the queue, at time t, with deadline in $[0,x]$. ξ_{0-}^N is the measure corresponding to the customers in queue at time 0.
- $\sigma_t^N = ess\inf(\xi_t^N)$-the left end of the support of the measure ξ_t^N.

Arrival, Service and Reneging

- $\alpha_t^N[0,x]$ The number of customers that have arrived in $(0,t]$ with deadlines in $[0,x]$.
- $\beta_t^{s,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that went into service during $(0,t]$.
- $\beta_t^{r,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that reneged during $(0,t]$, because their deadline elapsed before being admitted into the service.
- $\beta^N = \beta^{s,N} + \beta^{r,N}$.
- ρ_t^N the reneging count process, with values in $\mathbb{D}^{\uparrow}_{\mathbb{R}_+}$

$$\rho_t^N = \beta_t^{r,N}[0,\infty) = \beta_t^{r,N}[0,t] \quad \text{and} \quad \beta_t^{r,N}[0,x] = \rho_{t\wedge x}^N.$$

- the Cumulative Service Effort Process μ_t^N–allows for variable service rates.
- Service Process,$(\mathbf{S}(t))$- a renewal process with time between renewals distributed like the service times.

Arrival, Service and Reneging

- $\alpha_t^N[0,x]$ The number of customers that have arrived in $(0,t]$ with deadlines in $[0,x]$.
- $\beta_t^{s,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that went into service during $(0,t]$.
- $\beta_t^{r,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that reneged during $(0,t]$, because their deadline elapsed before being admitted into the service.
- $\beta^N = \beta^{s,N} + \beta^{r,N}$.
- ρ_t^N the reneging count process, with values in $\mathbb{D}^{\uparrow}_{\mathbb{R}_+}$

$$\rho_t^N = \beta_t^{r,N}[0,\infty) = \beta_t^{r,N}[0,t] \quad \text{and} \quad \beta_t^{r,N}[0,x] = \rho_{t\wedge x}^N.$$

- the Cumulative Service Effort Process μ_t^N–allows for variable service rates.
- Service Process,$(\mathbf{S}(t))$- a renewal process with time between renewals distributed like the service times.

Arrival, Service and Reneging

- $\alpha_t^N[0,x]$ The number of customers that have arrived in $(0,t]$ with deadlines in $[0,x]$.
- $\beta_t^{s,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that went into service during $(0,t]$.
- $\beta_t^{r,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that reneged during $(0,t]$, because their deadline elapsed before being admitted into the service.
- $\beta^N = \beta^{s,N} + \beta^{r,N}$.
- ρ_t^N the reneging count process, with values in $\mathbb{D}^{\uparrow}_{\mathbb{R}_+}$

$$\rho_t^N = \beta_t^{r,N}[0,\infty) = \beta_t^{r,N}[0,t] \quad \text{and} \quad \beta_t^{r,N}[0,x] = \rho_{t\wedge x}^N.$$

- the Cumulative Service Effort Process μ_t^N–allows for variable service rates.
- Service Process,$(\mathbf{S}(t))$- a renewal process with time between renewals distributed like the service times.

Arrival, Service and Reneging

- $\alpha_t^N[0,x]$ The number of customers that have arrived in $(0,t]$ with deadlines in $[0,x]$.
- $\beta_t^{s,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that went into service during $(0,t]$.
- $\beta_t^{r,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that reneged during $(0,t]$, because their deadline elapsed before being admitted into the service.
- $\beta^N = \beta^{s,N} + \beta^{r,N}$.
- ρ_t^N the reneging count process, with values in $\mathbb{D}^{\uparrow}_{\mathbb{R}_+}$

$$\rho_t^N = \beta_t^{r,N}[0,\infty) = \beta_t^{r,N}[0,t] \quad \text{and} \quad \beta_t^{r,N}[0,x] = \rho_{t\wedge x}^N.$$

- the Cumulative Service Effort Process μ_t^N–allows for variable service rates.
- Service Process,$(\mathbf{S}(t))$- a renewal process with time between renewals distributed like the service times.

Arrival, Service and Reneging

- $\alpha_t^N[0,x]$ The number of customers that have arrived in $(0,t]$ with deadlines in $[0,x]$.
- $\beta_t^{s,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that went into service during $(0,t]$.
- $\beta_t^{r,N}[0,x]$ The number of customers with deadlines in $[0,x]$ that reneged during $(0,t]$, because their deadline elapsed before being admitted into the service.
- $\beta^N = \beta^{s,N} + \beta^{r,N}$.
- ρ_t^N the reneging count process, with values in $\mathbb{D}^{\uparrow}_{\mathbb{R}_+}$

$$\rho_t^N = \beta_t^{r,N}[0,\infty) = \beta_t^{r,N}[0,t] \quad \text{and} \quad \beta_t^{r,N}[0,x] = \rho_{t\wedge x}^N.$$

- the Cumulative Service Effort Process μ_t^N–allows for variable service rates.
- Service Process,$(\mathbf{S}(t))$- a renewal process with time between renewals distributed like the service times.

The N system Dynamics

- Let B^N be a process with sample paths in $\mathbb{D}_{\mathbb{R}_+}$ defined as

$$B_t^N = \begin{cases} 1 & \text{if the server is busy at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

- The total number of customers in the system at time t, X_t^N, is given by

$$X_t^N = \xi_t^N[0,\infty) + B_t^N$$

- Let T_t^N, the cumulative service by time t,

$$T_t^N = \int_0^t B_s^N d\mu_s^N.$$

- the idleness ι_t^N, is given by

$$\iota_t^N = \int_0^t (1 - B_t^N) d\mu_t^N = \mu_t^N - T_t^N.$$

- the total number of jobs sent into service

$$\beta^{s,N}[0,\infty) = \mathbf{S}(T_t^N) + B_t^N.$$

The N system Dynamics

- Let B^N be a process with sample paths in $\mathbb{D}_{\mathbb{R}_+}$ defined as
$$B_t^N = \begin{cases} 1 & \text{if the server is busy at time } t, \\ 0 & \text{otherwise.} \end{cases}$$
- The total number of customers in the system at time t, X_t^N, is given by
$$X_t^N = \xi_t^N[0,\infty) + B_t^N$$
- Let T_t^N, the cumulative service by time t,
$$T_t^N = \int_0^t B_s^N d\mu_s^N.$$
- the idleness ι_t^N, is given by
$$\iota_t^N = \int_0^t (1 - B_t^N) d\mu_t^N = \mu_t^N - T_t^N.$$
- the total number of jobs sent into service
$$\beta^{s,N}[0,\infty) = \mathbf{S}(T_t^N) + B_t^N.$$

The N system equations

α^N arrival measure, β^N departure from the queue measure, ξ^N the queue measure, ρ^N reneging, μ^N cummulative service effort,ι^N idleness.

- $$\xi_t^N[0,x] = \left(\xi_{0-}^N[0,x] + \alpha_t^N[0,x] - (\mu_t^N + \rho_t^N)\right) + \beta_t^N(x,\infty) + \iota_t^N - e_t^N$$
- $$\xi_t^N[0,\infty) = \left(\xi_{0-}^N[0,\infty) + \alpha_t^N[0,\infty) - (\mu_t^N + \rho_t^N)\right) + \iota_t^N - e_t^N.$$
- Error term e_t^N
$$e_t^N = \beta^{s,N}[0,\infty) - T_t^N.$$

- EDF non preemptive priority

$$\int_0^\infty \xi_t^N[0,x]d\beta_t^N(x,\infty) = 0.$$

- Non idling

$$\int_0^\infty \xi_t^N[0,\infty)d\iota_t^N = 0.$$

- $\xi_t^N[0,t] = 0.$
- $\int_0^\infty 1_{\{\sigma_{t-}^N > t\}} d\rho_t^N = 0.$

Fluid Scaling and Assumptions

We scale by N.

- For a Borel set $B \subset \mathbb{R}_+$, and a measure $\nu^N =: \alpha_t^N, \xi_t^N, \xi_{0-}^N, \beta_t^{s,N}, \beta_t^{r,N}, \beta_t^N$, let $\bar{\nu}^N(B) = \frac{\nu^N(B)}{N}$.
- for a process $Y_t^N =: \mu_t^N, \iota_t^N, \rho_t^N, T_t^N, e_t^N$ in $\mathbb{D}_{\mathbb{R}_+}$, let $\bar{Y}_t^N = \frac{Y_t^N}{N}$.
- $\bar{\sigma}_t^N = \sigma_t^N$.

- **Assumption 1.** The measure valued random variable $\bar{\xi}_{0-}^N$ satisfies $\bar{\xi}_{0-}^N \Rightarrow \xi_{0-}$ where ξ_{0-} is a (non random) measure on $\mathbb{R}_+$.
- **Assumption 2.** The measure valued processes $\bar{\alpha}^N$ satisfy $\bar{\alpha}^N \Rightarrow \alpha$ where α is a (non random) measure valued process in $\mathbb{C}_{\mathcal{M}}^{\uparrow}$ and α has a (t,x) density.
- **Assumption 3.** the processes $\bar{\mu}^N$ satisfy $\bar{\mu}^N \Rightarrow \mu$ where μ is a (non random) member of $\mathbb{C}_{\mathbb{R}_+}^{\uparrow}$ with a density m that is locally bounded away from 0.

Fluid Scaling and Assumptions

We scale by N.

- For a Borel set $B \subset \mathbb{R}_+$, and a measure $\nu^N =: \alpha_t^N, \xi_t^N, \xi_{0-}^N, \beta_t^{s,N}, \beta_t^{r,N}, \beta_t^N$, let $\bar{\nu}^N(B) = \frac{\nu^N(B)}{N}$.
- for a process $Y_t^N =: \mu_t^N, \iota_t^N, \rho_t^N, T_t^N, e_t^N$ in $\mathbb{D}_{\mathbb{R}_+}$, let $\bar{Y}_t^N = \frac{Y_t^N}{N}$.
- $\bar{\sigma}_t^N = \sigma_t^N$.

- **Assumption 1.** The measure valued random variable $\bar{\xi}_{0-}^N$ satisfies $\bar{\xi}_{0-}^N \Rightarrow \xi_{0-}$ where ξ_{0-} is a (non random) measure on $\mathbb{R}_+$.
- **Assumption 2.** The measure valued processes $\bar{\alpha}^N$ satisfy $\bar{\alpha}^N \Rightarrow \alpha$ where α is a (non random) measure valued process in $\mathbb{C}_{\mathcal{M}}^{\uparrow}$ and α has a (t, x) density.
- **Assumption 3.** the processes $\bar{\mu}^N$ satisfy $\bar{\mu}^N \Rightarrow \mu$ where μ is a (non random) member of $\mathbb{C}_{\mathbb{R}_+}^{\uparrow}$ with a density m that is locally bounded away from 0.

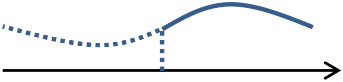
EDF earlier treatment

- B. Doytchinov, J. Lehoczky, S. Shreve, Real-time queues in heavy traffic with earliest-deadline-first queue discipline. Ann. Appl. Probab. 11 (2001)
- L. Decreusefond, P. Moyal, Fluid limit of a heavily loaded EDF queue with impatient customers, Markov Processes and Related fields 14 (2008)
- L. Kruk, J. Lehoczky, K. Ramanan, S. Shreve, Heavy-traffic analysis for EDF queues with reneging. Ann. Appl. Probab. 21 (2011)
- P. Moyal, On queues with impatience: stability, and the optimality of Earliest Deadline First, Preprint
- S. S. Panwar, D. Towsley, On the optimality of the STE rule for multiple server queues that serve customer with deadlines. Technical Report 8881, Dept. of Computer and Information Science, Univ. Massachusetts, Amherst.
- S. S. Panwar, D. Towsley, J. K. Wolf, Optimal scheduling policies for a class of queues with customer deadlines to the beginning of service. Journal of the Association for Computing Machinery, Vol.35
- A. Mandelbaum, P. Momcilovic, Personalized queues: the customer view, via least-patient-first routing. Preprint
- R. Atar, A. Biswas, H. Kaspi, Fluid limits of G/G/1+G queues under the non-preemptive earliest-deadline- first discipline. Preprint

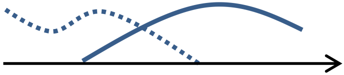
EDF/hard earlier treatment (contd)

Population density as a function of customer's deadline, x, at a given time, t.

- Dotted line: Customers that have arrived by time t and have been sent to service.
- Solid line: Customers that have arrived by time t and have not been sent to service.



separated populations



a generic situation

EDF/hard Comparison with earlier treatment

- Previous work on limit theorems for EDF-G/G/1 heavily relies on the fact that in the limit both arrival and service effort rates are constant.
- In those works, the Frontier Process plays a crucial role in the analysis. The frontier process $F^N(t)$ is the maximum of all the deadlines of jobs that have been in service by time t.
- When the arrival and service rates converge as $N \to \infty$ to a constant one can show that the limiting frontier process defines the limiting queue measure and further, that the situation depicted on the left side (where the dotted lines and solid lines are separated) prevails asymptotically and thus for the fluid model.
- This isn't true when the arrival and service rates are not constant.

EDF/hard Comparison with earlier treatment

- Previous work on limit theorems for EDF-G/G/1 heavily relies on the fact that in the limit both arrival and service effort rates are constant.
- In those works, the Frontier Process plays a crucial role in the analysis. The frontier process $F^N(t)$ is the maximum of all the deadlines of jobs that have been in service by time t.
- When the arrival and service rates converge as $N \to \infty$ to a constant one can show that the limiting frontier process defines the limiting queue measure and further, that the situation depicted on the left side (where the dotted lines and solid lines are separated) prevails asymptotically and thus for the fluid model.
- This isn't true when the arrival and service rates are not constant.

EDF/hard Comparison with earlier treatment

- Previous work on limit theorems for EDF-G/G/1 heavily relies on the fact that in the limit both arrival and service effort rates are constant.
- In those works, the Frontier Process plays a crucial role in the analysis. The frontier process $F^N(t)$ is the maximum of all the deadlines of jobs that have been in service by time t.
- When the arrival and service rates converge as $N \to \infty$ to a constant one can show that the limiting frontier process defines the limiting queue measure and further, that the situation depicted on the left side (where the dotted lines and solid lines are separated) prevails asymptotically and thus for the fluid model.
- This isn't true when the arrival and service rates are not constant.

EDF/hard Comparison with earlier treatment

- Previous work on limit theorems for EDF-G/G/1 heavily relies on the fact that in the limit both arrival and service effort rates are constant.
- In those works, the Frontier Process plays a crucial role in the analysis. The frontier process $F^N(t)$ is the maximum of all the deadlines of jobs that have been in service by time t.
- When the arrival and service rates converge as $N \to \infty$ to a constant one can show that the limiting frontier process defines the limiting queue measure and further, that the situation depicted on the left side (where the dotted lines and solid lines are separated) prevails asymptotically and thus for the fluid model.
- This isn't true when the arrival and service rates are not constant.

One-dimensional Skorohod problem

Problem

Given data $\psi \in \mathbb{D}_{\mathbb{R}}$*, find a pair* $(\varphi, \eta) \in \mathbb{D}_{\mathbb{R}} \times \mathbb{D}_{\mathbb{R}}^{\uparrow}$ *such that* $\varphi = \psi + \eta$ *and* $\varphi(t) \geq 0$ *for all* t*, and* $\varphi = 0$ $d\eta$*-a.e.*

Solution

It is well-known that the unique solution is given by

$$\varphi = \psi - \inf_{[0,\cdot]}(\psi \wedge 0), \qquad \eta = -\inf_{[0,\cdot]}(\psi \wedge 0).$$

We call $\varphi = \Gamma_1(\psi)$ and $\eta = \Gamma_2(\psi)$.

One-dimensional Skorohod problem

Problem

Given data $\psi \in \mathbb{D}_{\mathbb{R}}$*, find a pair* $(\varphi, \eta) \in \mathbb{D}_{\mathbb{R}} \times \mathbb{D}_{\mathbb{R}}^{\uparrow}$ *such that* $\varphi = \psi + \eta$ *and* $\varphi(t) \geq 0$ *for all* t*, and* $\varphi = 0$ $d\eta$*-a.e.*

Solution

It is well-known that the unique solution is given by

$$\varphi = \psi - \inf_{[0,\cdot]}(\psi \wedge 0), \qquad \eta = -\inf_{[0,\cdot]}(\psi \wedge 0).$$

We call $\varphi = \Gamma_1(\psi)$ and $\eta = \Gamma_2(\psi)$.

One-dimensional Skorohod problem

Problem

Given data $\psi \in \mathbb{D}_{\mathbb{R}}$*, find a pair* $(\varphi, \eta) \in \mathbb{D}_{\mathbb{R}} \times \mathbb{D}_{\mathbb{R}}^{\uparrow}$ *such that* $\varphi = \psi + \eta$ *and* $\varphi(t) \geq 0$ *for all* t*, and* $\varphi = 0$ $d\eta$*-a.e.*

Solution

It is well-known that the unique solution is given by

$$\varphi = \psi - \inf_{[0,\cdot]}(\psi \wedge 0), \qquad \eta = -\inf_{[0,\cdot]}(\psi \wedge 0).$$

We call $\varphi = \Gamma_1(\psi)$ and $\eta = \Gamma_2(\psi)$.

Infinite Dimensional Skorokhod Map-IDSM-
The Skorokhod map applied to measure valued processes

Notation needed:

- $\mathcal{M}$ = finite Borel measures on $\mathbb{R}_+$ with the topology of weak convergence.
- $\mathbb{D}_{\mathcal{M}}$ = cadlag functions $\mathbb{R}_+ \to \mathcal{M}$.
- $\mathbb{D}^{\uparrow}_{\mathcal{M}}$ = those members of $\mathbb{D}_{\mathcal{M}}$ that are nondecreasing (i.e., $\xi_t - \xi_s \in \mathcal{M}$ for $t > s$).
- $\mathbb{C}^{\uparrow}_{\mathcal{M}}$ = those members of $\mathbb{C}_{\mathcal{M}}$ that are nondecreasing.
- $\mathbb{C}^{\uparrow,0}_{\mathcal{M}} = \{\zeta \in \mathbb{C}^{\uparrow}_{\mathcal{M}}$: for each t, the measure ζ_t has no atoms$\}$.

Infinite Dimensional Skorokhod Map-IDSM-
The Skorokhod map applied to measure valued processes

Notation needed:

- $\mathcal{M}$ = finite Borel measures on $\mathbb{R}_+$ with the topology of weak convergence.
- $\mathbb{D}_{\mathcal{M}}$ = cadlag functions $\mathbb{R}_+ \to \mathcal{M}$.
- $\mathbb{D}^{\uparrow}_{\mathcal{M}}$ = those members of $\mathbb{D}_{\mathcal{M}}$ that are nondecreasing (i.e., $\xi_t - \xi_s \in \mathcal{M}$ for $t > s$).
- $\mathbb{C}^{\uparrow}_{\mathcal{M}}$ = those members of $\mathbb{C}_{\mathcal{M}}$ that are nondecreasing.
- $\mathbb{C}^{\uparrow,0}_{\mathcal{M}} = \{\zeta \in \mathbb{C}^{\uparrow}_{\mathcal{M}}$: for each t, the measure ζ_t has no atoms$\}$.

Infinite Dimensional Skorokhod Problem for EDF/soft

Problem

Given data $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *find* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *such that for each* $x \in [0, \infty)$,

1. $\xi[0,x] = \alpha[0,x] - \mu + \beta(x,\infty) + \iota$,
2. $\xi[0,\infty) = \alpha[0,\infty) - \mu + \iota$,
3. $\xi[0,x] = 0 \; d\beta(x,\infty)$-*a.e.*,
4. $\xi[0,\infty) = 0 \; d\iota$-*a.e.*

Considering first equations 2. and 4. it is clear that one can define

$$(\xi[0,\infty), \iota) = \Gamma(\alpha[0,\infty) - \mu) \qquad \iota = \Gamma_2(\alpha[0,\infty) - \mu) \qquad (1)$$

With ι at hand, considering for each $x \in [0,\infty)$ equations 1.,3.

$$\xi_t[0,x] = \Gamma_1(\alpha[0,x] - \mu + \iota), \qquad \beta(x,\infty) = \Gamma_2(\alpha[0,x] - \mu + \iota) \quad (2)$$

Infinite Dimensional Skorokhod Problem for EDF/soft

Problem

Given data $(\alpha,\mu)\in\mathbb{D}^{\uparrow}_{\mathcal{M}}\times\mathbb{D}^{\uparrow}_{\mathbb{R}}$ *find* $(\xi,\beta,\iota)\in\mathbb{D}_{\mathcal{M}}\times\mathbb{D}^{\uparrow}_{\mathcal{M}}\times\mathbb{D}^{\uparrow}_{\mathbb{R}}$ *such that for each* $x\in[0,\infty)$*,*

1. $\xi[0,x]=\alpha[0,x]-\mu+\beta(x,\infty)+\iota$,
2. $\xi[0,\infty)=\alpha[0,\infty)-\mu+\iota$,
3. $\xi[0,x]=0\ d\beta(x,\infty)$-*a.e.*,
4. $\xi[0,\infty)=0\ d\iota$-*a.e.*

Considering first equations 2. and 4. it is clear that one can define

$$(\xi[0,\infty),\iota)=\Gamma(\alpha[0,\infty)-\mu)\qquad \iota=\Gamma_2(\alpha[0,\infty)-\mu)\qquad (1)$$

With ι at hand, considering for each $x\in[0,\infty)$ equations 1.,3.

$$\xi_t[0,x]=\Gamma_1(\alpha[0,x]-\mu+\iota),\qquad \beta(x,\infty)=\Gamma_2(\alpha[0,x]-\mu+\iota)\quad (2)$$

Infinite Dimensional Skorokhod Problem for EDF/soft

Problem

Given data $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *find* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *such that for each* $x \in [0, \infty)$*,*

1. $\xi[0,x] = \alpha[0,x] - \mu + \beta(x,\infty) + \iota$,
2. $\xi[0,\infty) = \alpha[0,\infty) - \mu + \iota$,
3. $\xi[0,x] = 0 \; d\beta(x,\infty)$*-a.e.*,
4. $\xi[0,\infty) = 0 \; d\iota$*-a.e.*

Considering first equations 2. and 4. it is clear that one can define

$$(\xi[0,\infty), \iota) = \Gamma(\alpha[0,\infty) - \mu) \qquad \iota = \Gamma_2(\alpha[0,\infty) - \mu) \qquad (1)$$

With ι at hand, considering for each $x \in [0,\infty)$ equations 1.,3.

$$\xi_t[0,x] = \Gamma_1(\alpha[0,x] - \mu + \iota), \qquad \beta(x,\infty) = \Gamma_2(\alpha[0,x] - \mu + \iota) \quad (2)$$

Infinite Dimensional Skorokhod Problem for EDF/soft

Problem

Given data $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *find* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *such that for each* $x \in [0, \infty)$,

1. $\xi[0, x] = \alpha[0, x] - \mu + \beta(x, \infty) + \iota$,
2. $\xi[0, \infty) = \alpha[0, \infty) - \mu + \iota$,
3. $\xi[0, x] = 0 \; d\beta(x, \infty)$*-a.e.*,
4. $\xi[0, \infty) = 0 \; d\iota$*-a.e.*

Considering first equations 2. and 4. it is clear that one can define

$$(\xi[0, \infty), \iota) = \Gamma(\alpha[0, \infty) - \mu) \qquad \iota = \Gamma_2(\alpha[0, \infty) - \mu) \qquad (1)$$

With ι at hand, considering for each $x \in [0, \infty)$ equations 1.,3.

$$\xi_t[0, x] = \Gamma_1(\alpha[0, x] - \mu + \iota), \qquad \beta(x, \infty) = \Gamma_2(\alpha[0, x] - \mu + \iota) \quad (2)$$

Infinite Dimensional Skorokhod Problem for EDF/soft

Problem

Given data $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *find* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *such that for each* $x \in [0, \infty)$*,*

1. $\xi[0,x] = \alpha[0,x] - \mu + \beta(x,\infty) + \iota$,
2. $\xi[0,\infty) = \alpha[0,\infty) - \mu + \iota$,
3. $\xi[0,x] = 0 \; d\beta(x,\infty)$*-a.e.,*
4. $\xi[0,\infty) = 0 \; d\iota$*-a.e.*

Considering first equations 2. and 4. it is clear that one can define

$$(\xi[0,\infty), \iota) = \Gamma(\alpha[0,\infty) - \mu) \qquad \iota = \Gamma_2(\alpha[0,\infty) - \mu) \qquad (1)$$

With ι at hand, considering for each $x \in [0,\infty)$ equations 1.,3.

$$\xi_t[0,x] = \Gamma_1(\alpha[0,x] - \mu + \iota), \qquad \beta(x,\infty) = \Gamma_2(\alpha[0,x] - \mu + \iota) \quad (2)$$

Infinite Dimensional Skorokhod Problem for EDF/soft

Problem

Given data $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *find* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *such that for each* $x \in [0, \infty)$,

1. $\xi[0, x] = \alpha[0, x] - \mu + \beta(x, \infty) + \iota$,
2. $\xi[0, \infty) = \alpha[0, \infty) - \mu + \iota$,
3. $\xi[0, x] = 0 \; d\beta(x, \infty)$-*a.e.*,
4. $\xi[0, \infty) = 0 \; d\iota$-*a.e.*

Considering first equations 2. and 4. it is clear that one can define

$$(\xi[0, \infty), \iota) = \Gamma(\alpha[0, \infty) - \mu) \qquad \iota = \Gamma_2(\alpha[0, \infty) - \mu) \tag{1}$$

With ι at hand, considering for each $x \in [0, \infty)$ equations 1.,3.

$$\xi_t[0, x] = \Gamma_1(\alpha[0, x] - \mu + \iota), \qquad \beta(x, \infty) = \Gamma_2(\alpha[0, x] - \mu + \iota) \tag{2}$$

Infinite Dimensional Skorokhod Problem for EDF/soft

Problem

Given data $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *find* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}}$ *such that for each* $x \in [0, \infty)$*,*

1. $\xi[0, x] = \alpha[0, x] - \mu + \beta(x, \infty) + \iota$,
2. $\xi[0, \infty) = \alpha[0, \infty) - \mu + \iota$,
3. $\xi[0, x] = 0 \ d\beta(x, \infty)$*-a.e.,*
4. $\xi[0, \infty) = 0 \ d\iota$*-a.e.*

Considering first equations 2. and 4. it is clear that one can define

$$(\xi[0, \infty), \iota) = \Gamma(\alpha[0, \infty) - \mu) \qquad \iota = \Gamma_2(\alpha[0, \infty) - \mu) \tag{1}$$

With ι at hand, considering for each $x \in [0, \infty)$ equations 1.,3.

$$\xi_t[0, x] = \Gamma_1(\alpha[0, x] - \mu + \iota), \qquad \beta(x, \infty) = \Gamma_2(\alpha[0, x] - \mu + \iota) \tag{2}$$

The Solution to the IDSP

Theorem

- *For every* $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$ *there exists a unique* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$ *such that (1-4) hold.*

- *Let* Θ *denote the corresponding map from* $\mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$ *to* $\mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$. *Then* Θ *is measurable and continuous on* $\mathbb{C}^{\uparrow,0}_{\mathcal{M}} \times \mathbb{C}^{\uparrow}_{\mathbb{R}_+}$.

Let $\mathit{\Theta}$ = the solution map. Then the **fluid model for EDF/soft** reads

$$(\xi, \beta, \iota) = \mathit{\Theta}(\alpha, \mu)$$

The Solution to the IDSP

Theorem

- *For every* $(\alpha, \mu) \in \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$ *there exists a unique* $(\xi, \beta, \iota) \in \mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$ *such that (1-4) hold.*
- *Let* Θ *denote the corresponding map from* $\mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$ *to* $\mathbb{D}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathcal{M}} \times \mathbb{D}^{\uparrow}_{\mathbb{R}_+}$. *Then* Θ *is measurable and continuous on* $\mathbb{C}^{\uparrow,0}_{\mathcal{M}} \times \mathbb{C}^{\uparrow}_{\mathbb{R}_+}$.

Let $\mathit{\Theta}$ = the solution map. Then the **fluid model for EDF/soft** reads

$$(\xi, \beta, \iota) = \mathit{\Theta}(\alpha, \mu)$$

EDF/hard via the IDSM

- α, μ, ξ, ι have the same meaning as before.
- $\beta_t(B)$ = mass of customers with deadline in B, that by time t have left the queue: either by transferring to the server **or by reneging**.
- $\rho \in \mathbb{D}^{\uparrow}_{\mathbb{R}}$. ρ_t = amount of mass that has left the system by reneging until time t.

EDF/hard via the IDSM

- α, μ, ξ, ι have the same meaning as before.
- $\beta_t(B)$ = mass of customers with deadline in B, that by time t have left the queue: either by transferring to the server **or by reneging**.
- $\rho \in \mathbb{D}^{\uparrow}_{\mathbb{R}}$. ρ_t = amount of mass that has left the system by reneging until time t.

EDF/hard via the IDSM

- α, μ, ξ, ι have the same meaning as before.
- $\beta_t(B) =$ mass of customers with deadline in B, that by time t have left the queue: either by transferring to the server **or by reneging**.
- $\rho \in \mathbb{D}_{\mathbb{R}}^{\uparrow}$. $\rho_t =$ amount of mass that has left the system by reneging until time t.

EDF/hard via the IDSM

- α, μ, ξ, ι have the same meaning as before.
- $\beta_t(B) =$ mass of customers with deadline in B, that by time t have left the queue: either by transferring to the server **or by reneging**.
- $\rho \in \mathbb{D}^{\uparrow}_{\mathbb{R}}$. $\rho_t =$ amount of mass that has left the system by reneging until time t.

Form a set of equations:

1. $\xi[0,x] = \alpha[0,x] - (\mu + \rho) + \beta(x,\infty) + \iota$,
2. $\xi[0,\infty) = \alpha[0,\infty) - (\mu + \rho) + \iota$,
3. $\xi[0,x] = 0 \; d\beta(x,\infty)$-a.e.
4. $\xi[0,\infty) = 0 \; d\iota$-a.e.

So that

$$\begin{cases} (i) & (\xi,\beta,\iota) = \Theta(\alpha, \mu+\rho). \\ (ii) & \xi_t[0,t) = 0, \quad \text{for every } t > 0. \end{cases}$$

- Consider the partial order $\rho \leq \tilde{\rho}$ iff $\rho_t \leq \tilde{\rho}_t$ for every t.
- **Fluid model for EDF/hard** is given as: Minimal solution of (i)+(ii).
- **Fact:** There exists a unique minimal solution.

Form a set of equations:

1. $\xi[0,x] = \alpha[0,x] - (\mu + \rho) + \beta(x,\infty) + \iota$,
2. $\xi[0,\infty) = \alpha[0,\infty) - (\mu + \rho) + \iota$,
3. $\xi[0,x] = 0 \ d\beta(x,\infty)$-a.e.
4. $\xi[0,\infty) = 0 \ d\iota$-a.e.

So that

$$\begin{cases} (i) & (\xi,\beta,\iota) = \Theta(\alpha,\mu+\rho). \\ (ii) & \xi_t[0,t) = 0, \quad \text{for every } t > 0. \end{cases}$$

- Consider the partial order $\rho \leq \tilde{\rho}$ iff $\rho_t \leq \tilde{\rho}_t$ for every t.
- **Fluid model for EDF/hard** is given as: Minimal solution of (i)+(ii).
- **Fact:** There exists a unique minimal solution.

Form a set of equations:

1. $\xi[0,x] = \alpha[0,x] - (\mu + \rho) + \beta(x,\infty) + \iota$,
2. $\xi[0,\infty) = \alpha[0,\infty) - (\mu + \rho) + \iota$,
3. $\xi[0,x] = 0 \; d\beta(x,\infty)$-a.e.
4. $\xi[0,\infty) = 0 \; d\iota$-a.e.

So that

$$\begin{cases} (i) & (\xi,\beta,\iota) = \Theta(\alpha,\mu+\rho). \\ (ii) & \xi_t[0,t) = 0, \quad \text{for every } t > 0. \end{cases}$$

- Consider the partial order $\rho \leq \tilde{\rho}$ iff $\rho_t \leq \tilde{\rho}_t$ for every t.
- **Fluid model for EDF/hard** is given as: Minimal solution of (i)+(ii).
- **Fact:** There exists a unique minimal solution.

Form a set of equations:

1. $\xi[0,x] = \alpha[0,x] - (\mu + \rho) + \beta(x,\infty) + \iota$,
2. $\xi[0,\infty) = \alpha[0,\infty) - (\mu + \rho) + \iota$,
3. $\xi[0,x] = 0 \ d\beta(x,\infty)$-a.e.
4. $\xi[0,\infty) = 0 \ d\iota$-a.e.

So that

$$\begin{cases} (i) & (\xi,\beta,\iota) = \Theta(\alpha, \mu + \rho). \\ (ii) & \xi_t[0,t) = 0, \quad \text{for every } t > 0. \end{cases}$$

- Consider the partial order $\rho \leq \tilde{\rho}$ iff $\rho_t \leq \tilde{\rho}_t$ for every t.
- **Fluid model for EDF/hard** is given as: Minimal solution of (i)+(ii).
- **Fact:** There exists a unique minimal solution.

Fluid model - an alternative characterization

$$\begin{cases} (i) & (\xi, \beta, \iota) = \Theta(\alpha, \mu + \rho) \\ (ii) & \xi_t[0, t) = 0, \quad \text{for every } t > 0 \\ (iii) & \sigma_t = t \quad d\rho\text{-a.e., where for } t \geq 0,\ \sigma_t = ess \inf \xi_t. \end{cases}$$

Theorem

Assume that α has (t, x)-density, μ has a density locally bounded away from 0 and ξ_{0-}, the initial measure, has no atoms .Then

- *there exists a unique solution $(\xi, \beta, \iota, \rho)$ to (i)+(ii)+(iii),*
- *it is equal to the minimal solution of (i)+(ii).*

This provides an additional formulation of the fluid model:
Fluid model has minimality properties.

Fluid model - an alternative characterization

$$\begin{cases} (i) & (\xi, \beta, \iota) = \Theta(\alpha, \mu + \rho) \\ (ii) & \xi_t[0, t) = 0, \quad \text{for every } t > 0 \\ (iii) & \sigma_t = t \quad d\rho\text{-a.e., where for } t \geq 0,\ \sigma_t = ess \inf \xi_t. \end{cases}$$

Theorem

Assume that α has (t, x)-density, μ has a density locally bounded away from 0 and ξ_{0-}, the initial measure, has no atoms . Then

- *there exists a unique solution $(\xi, \beta, \iota, \rho)$ to (i)+(ii)+(iii),*
- *it is equal to the minimal solution of (i)+(ii).*

This provides an additional formulation of the fluid model:
Fluid model has minimality properties.

Fluid model - an alternative characterization

$$\begin{cases} (i) & (\xi, \beta, \iota) = \Theta(\alpha, \mu + \rho) \\ (ii) & \xi_t[0, t) = 0, \quad \text{for every } t > 0 \\ (iii) & \sigma_t = t \quad d\rho\text{-a.e., where for } t \geq 0,\ \sigma_t = ess \inf \xi_t. \end{cases}$$

Theorem

Assume that α has (t, x)-density, μ has a density locally bounded away from 0 and ξ_{0-}, the initial measure, has no atoms .Then

- *there exists a unique solution $(\xi, \beta, \iota, \rho)$ to (i)+(ii)+(iii),*
- *it is equal to the minimal solution of (i)+(ii).*

This provides an additional formulation of the fluid model:
Fluid model has minimality properties.

Fluid model - an alternative characterization

$$\begin{cases} (i) & (\xi, \beta, \iota) = \Theta(\alpha, \mu + \rho) \\ (ii) & \xi_t[0, t) = 0, \quad \text{for every } t > 0 \\ (iii) & \sigma_t = t \quad d\rho\text{-a.e., where for } t \geq 0,\ \sigma_t = ess \inf \xi_t. \end{cases}$$

Theorem

Assume that α has (t, x)-density, μ has a density locally bounded away from 0 and ξ_{0-}, the initial measure, has no atoms .Then

- *there exists a unique solution $(\xi, \beta, \iota, \rho)$ to (i)+(ii)+(iii),*
- *it is equal to the minimal solution of (i)+(ii).*

This provides an additional formulation of the fluid model:
Fluid model has minimality properties.

Fluid model - an alternative characterization

$$\begin{cases} (i) & (\xi, \beta, \iota) = \Theta(\alpha, \mu + \rho) \\ (ii) & \xi_t[0, t) = 0, \quad \text{for every } t > 0 \\ (iii) & \sigma_t = t \quad d\rho\text{-a.e., where for } t \geq 0,\ \sigma_t = ess \inf \xi_t. \end{cases}$$

Theorem

Assume that α has (t, x)-density, μ has a density locally bounded away from 0 and ξ_{0-}, the initial measure, has no atoms .Then

- *there exists a unique solution $(\xi, \beta, \iota, \rho)$ to (i)+(ii)+(iii),*
- *it is equal to the minimal solution of (i)+(ii).*

This provides an additional formulation of the fluid model:
Fluid model has minimality properties.

Convergence of the scaled N- System

Recall the assumptions we made for the EDF/hard model:

- **Assumption 1.** The measure valued random variable $\bar{\xi}_{0-}^N$ satisfies $\bar{\xi}_{0-}^N \Rightarrow \xi_{0-}$ where ξ_{0-} is a (non random) measure on $\mathbb{R}_+$.
- **Assumption 2.** The measure valued processes $\bar{\alpha}^N$ satisfies $\bar{\alpha}^N \Rightarrow \alpha$ where α is a (non random) measure valued process in $\mathbb{C}_{\mathcal{M}}^{\uparrow}$ and α has a (t, x) density.
- **Assumption 3.** the processes $\bar{\mu}^N$ satisfy $\bar{\mu}^N \Rightarrow \mu$ where μ is a (non random) member of $\mathbb{C}_{\mathbb{R}_+}^{\uparrow}$ with a density m that is locally bounded away from 0.

Theorem

Let Assumptions 1,2 and 3 hold. Then $\left(\bar{\xi}^N, \bar{\beta}^N, \bar{\iota}^N, \bar{\rho}^N, \bar{e}^N\right) \Rightarrow (\xi, \beta, \iota, \rho, 0)$. $(\xi, \beta, \iota, \rho)$ *is the unique solution ofthe IDSP for* (ξ_{0-}, α, μ). *It lies in* $\mathbb{C}_{\mathcal{M}} \times \mathbb{C}_{\mathcal{M}}^{\uparrow} \times \mathbb{C}_{\mathbb{R}_+}^{\uparrow} \times \mathbb{C}_{\mathbb{R}_+}^{\uparrow}$, *and the convergence is u.o.c.*

Previous results

- A two sided infinite dimensional Skorokhod map was introduced in the work of Kruk, Lehotzky, Ramanan and Shreve 2011 on the EDF queue.
- In that paper it was shown that, under constant arrival and service rate, the mount of work that gets late is minimal among all service protocols.

Extensions and Other uses of the IDSM

- EDF for many server queues-much harder because there μ is also a part of the solution rather than the given data. Solved with another IDSM.
- Applied the IDSM to Queues that prioritize according to the Shortest Remaining Processing Time First(SRPT).
- Hope that the IDSM will find use outside of queueing theory.

THANK YOU