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Normal approximation (reminder)

Simple random walk (1-dim): $S_n = X_1 + \dots + X_n$, X_k iid random signs.

$\mathbb{P}\big(\frac{1}{\sqrt{n}} S_n > c\big) \to \mathbb{P}\big(\zeta > c\big)$, $\quad \zeta \sim N(0,1)$. (fair coin)

$\mathbb{P}\big(\frac{1}{\sqrt{n}} S_n > c_n\big) \sim \mathbb{P}\big(\zeta > c_n\big)$? Yes, if $c_n = o(n^{1/4})$.

Asymmetry harms between $n^{1/6}$ and $n^{1/4}$, curtosis between $n^{1/4}$ and $n^{3/10}$, ...

$\log \mathbb{P}\big(\frac{1}{\sqrt{n}} S_n > c_n\big) \sim \log \mathbb{P}\big(\zeta > c_n\big)$? Yes, if $0 < c_n = o(\sqrt{n}\,)$. MDP

No, if $c_n = \sqrt{n}$; the end of universality. LDP

$\log \mathbb{E} \exp \lambda \frac{1}{\sqrt{n}} S_n \to \log \mathbb{E} \exp \lambda\zeta = \frac{1}{2}\lambda^2$. cumulant generating function

$\log \mathbb{E} \exp \lambda_n S_n \sim \frac{n}{2}\lambda_n^2$? Yes, if $\lambda_n \to 0$,

since $\log \mathbb{E} \exp \lambda_n S_n = n \log \cosh \lambda_n = n \log(1 + \frac{1}{2}\lambda_n^2 + \dots)$

Arbitrary S_n: If $\frac{1}{n} \log \mathbb{E} \exp \lambda_n S_n \sim \frac{1}{2}\lambda_n^2$ for $\lambda_n \to 0$ LinearResponceP

then $\log \mathbb{P}\big(S_n > c_n\sqrt{n}\big) \sim \log \mathbb{P}\big(\zeta > c_n\big)$ for $0 < c_n = o(\sqrt{n}\,)$;

that is, $\mathbb{P}\big(S_n > c\sqrt{n}\big) \to \mathbb{P}\big(\zeta > c\big)$ CLT

and $\log \mathbb{P}\big(S_n > c_n\sqrt{n}\big) \sim -\frac{1}{2}c_n^2$ for $c_n \to +\infty$, $c_n = o(\sqrt{n}\,)$. MDP

LRP $\Longrightarrow$ MDP by Gärtner(-Ellis); but MDP $\not\Longrightarrow$ LRP.

Random entire function: $\psi(z)=\sum_{k=0}^{\infty}\frac{\zeta_k z^k}{\sqrt{k!}}$, $z\in\mathbb{C}$,

$\zeta_0,\zeta_1,\dots$ are independent standard complex-Gaussian random variables.

Random field X on $\mathbb{R}^2$, invariant in distribution under shifts and rotations:

$$X(\operatorname{Re} z,\operatorname{Im} z)=\log|\psi(z)|-\tfrac12|z|^2+\tfrac12\gamma_{\text{Euler}};$$ ΔX = zeroes

$\mathbb{E}\,X(\cdot,\cdot)=0$. Then large-scale behavior of zeroes; linear statistic

$$\sum_{z:\psi(z)=0} h(\operatorname{Re} z,\operatorname{Im} z)-\frac{1}{\pi}\iint_{\mathbb{R}^2} h(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2=\frac{1}{2\pi}\iint_{\mathbb{R}^2} X(t_1,t_2)\Delta h(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2$$

for every compactly supported C^2-function $h:\mathbb{R}^2\to\mathbb{R}$. test function

Theorem. For every compactly supported bounded measurable $f:\mathbb{R}^2\to\mathbb{R}$,

$$\lim_{\substack{r\to\infty\\ \lambda\log^2 r\to 0}}\frac{1}{r^2\lambda^2}\log\mathbb{E}\exp\lambda\iint_{\mathbb{R}^2} f\Big(\frac{t_1}{r},\frac{t_2}{r}\Big)X_{t_1,t_2}\,\mathrm{d}t_1\mathrm{d}t_2=\frac{\sigma^2}{2}\|f\|^2_{L_2(\mathbb{R}^2)}.$$ rescaling

$\sigma=\frac12\sqrt{\pi\zeta(3)}$ predicted: Forrester & Honner 1999, confirmed: Nazarov & Sodin 2011.

Corollary. LRP $\Longrightarrow$ MDP & CLT

$$\lim_{\substack{r\to\infty,c\to\infty\\ (c\log^2 r)/r\to 0}}\frac{1}{c^2}\log\mathbb{P}\bigg(\iint f\Big(\frac{t_1}{r},\frac{t_2}{r}\Big)X_{t_1,t_2}\,\mathrm{d}t_1\mathrm{d}t_2\ge c\sigma\|f\|r\bigg)=-\frac12,$$

and the distribution of $r^{-1}\iint f\big(\frac{t_1}{r},\frac{t_2}{r}\big)X_{t_1,t_2}\,\mathrm{d}t_1\mathrm{d}t_2$ converges (as $r\to\infty$) to the normal distribution $N(0,\sigma^2\|f\|^2)$. (note log. gap MDP–LDP)

One-dimensional counterpart

Over $\mathbb{R}^2$: $X(\operatorname{Re} z, \operatorname{Im} z) - \frac{1}{2}\gamma_{\text{Euler}} = \log|\underbrace{\psi(z)e^{-|z|^2/2}}_{\xi(\operatorname{Re} z, \operatorname{Im} z)}|$; $\mathbb{E}\,\psi(z_1)\overline{\psi(z_2)} = \exp(z_1\overline{z_2})$;

$\mathbb{E}\,\xi(s)\overline{\xi(t)} = \exp\big(\mathrm{i}\,\underbrace{(s \wedge t)}_{=0 \text{ for collinear } s,t} - \frac{1}{2}|s-t|^2\big)$.

Over $\mathbb{R}$: stationary non-Gaussian random process $X_t = \log|\xi(t)| + \frac{1}{2}\gamma_{\text{Euler}}$ (real-valued), where ξ is a stationary complex-Gaussian random process with covariance function $t \mapsto \exp(-t^2/2)$. X is splittable.

Splittable processes

Definition. A random process $X = (X_t)_{t\in\mathbb{R}}$ is *splittable*, if there exists $c > 0$ such that for every $t \in \mathbb{R}$,

first, $\log \mathbb{E}\exp(c|X_t|) \le 1$ and $\mathbb{E}\,X_t = 0$; second, there exists (on some probability space) a triple of random processes X^0, X^-, X^+ such that

(a) the two processes X^-, X^+ are independent;

(b) the four processes X, X^0, X^-, X^+ are identically distributed;

(c) $\log \mathbb{E}\exp\big(c\int_{-\infty}^t |X_s^- - X_s^0|\,\mathrm{d}s + c\int_t^\infty |X_s^+ - X_s^0|\,\mathrm{d}s\big) \le 1$.

The number "1" may be replaced with any other positive number.

SPLIT $\Longrightarrow$ LRP $\Longrightarrow$ MDP & CLT

THEOREM. For every splittable $(X_t)_{t\in\mathbb{R}}$ there exists $C>0$ such that for all $a,b\in\mathbb{R}$ such that $b-a\ge 2$ and all $f\in L_\infty(a,b)$ such that $C\|f\|_\infty \log(b-a)\le 1$ the random variable

$$S=\int_a^b f(t)X_t\,\mathrm{d}t$$

satisfies

$$\frac{1}{b-a}\Big|\log\mathbb{E}\exp S-\frac12\mathbb{E}\,S^2\Big|\le C\|f\|_\infty^2\big(\|f\|_\infty\log(b-a)\big)^{1/3}.$$

Thus, for $f\in L_\infty(0,1)$ the random variables

$$S_n=\int_0^n f\Big(\frac{t}{n}\Big)X_t\,\mathrm{d}t$$

satisfy $\frac1n|\log\mathbb{E}\exp\lambda_n S_n-\frac12\lambda_n^2\mathbb{E}\,S_n^2|\le C\lambda_n^2\|f\|_\infty^2(\lambda_n\|f\|_\infty\log n)^{1/3}$ for $\lambda_n\log n\to 0$; assuming also $\mathbb{E}\,S_n^2\sim n$ we get (note log. gap MDP–LDP)

$$\frac1n\log\mathbb{E}\exp\lambda_n S_n\sim\frac12\lambda_n^2 \qquad \text{for } \lambda_n\log n\to 0\,, \qquad \text{LRP}$$

$$\log\mathbb{P}\big(S_n>c_n\sqrt n\big)\sim-\frac12 c_n^2 \qquad \text{for } c_n\to+\infty, c_n=o\Big(\frac{\sqrt n}{\log n}\Big)\,, \qquad \text{MDP}$$

$$\mathbb{P}\big(S_n>c\sqrt n\big)\to\mathbb{P}\big(\zeta>c\big) \qquad \text{for } c\in\mathbb{R}\,. \qquad \text{CLT}$$

CLT fails for unbounded f: Nazarov & Sodin 2011, “Fluctuations…” Sect. 5.2.

The same LRP implies also MDP (and CLT) for joint distributions. Like this: if $f, g \in L_\infty(0,1)$ are in $L_2(0,1)$ orthogonal, of length 1, then

$$\log \mathbb{P}\left(\frac{1}{c_n\sqrt{n}}\big(S_n(f), S_n(g)\big) \in U \right) \sim -\frac{1}{2}\, c_n^2 \inf_{(x,y)\in U} (x^2+y^2)$$

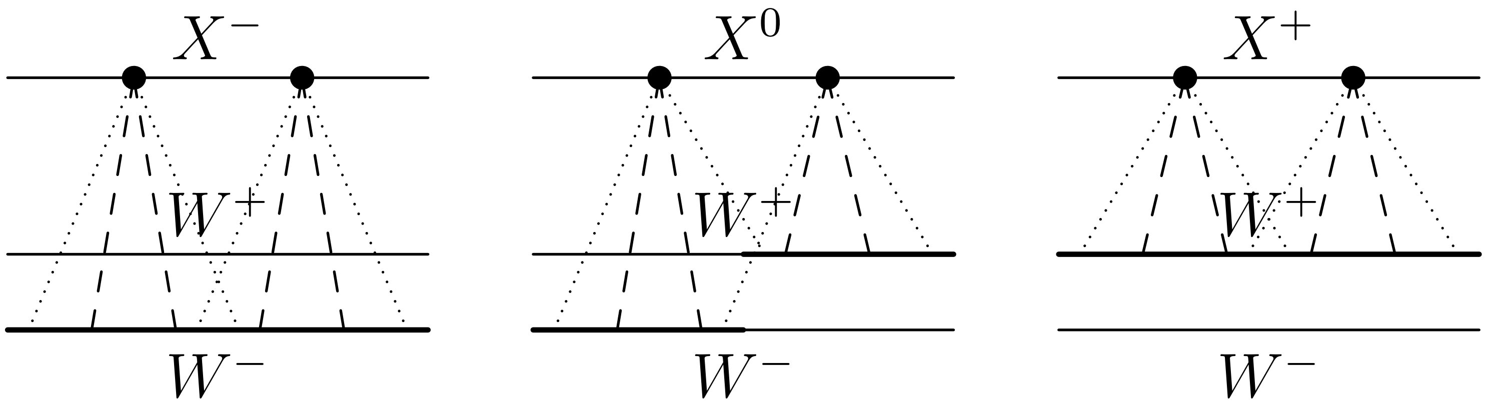
for open $U \subset \mathbb{R}^2$. Least unlikely

About proofs: SPLIT $\Longrightarrow$ LRP

$\Lambda_n(\lambda) = \log \mathbb{E} \exp \lambda \int_0^n X_t \,\mathrm{d}t$; estimate Λ_{2n} in terms of Λ_n via Hölder's inequality; chain: $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_8, \dots$; choose Hölder exponent for Λ_{2^k} as $\approx 1 + \max(2^{-k/2}, |\lambda|)$. Asymptotically quadratic, since generally $|\Lambda(\lambda) - \frac{1}{2}\Lambda''(0)\lambda^2| \le 0.35\big(\frac{|\lambda|}{c-|\lambda|}\big)^3\big(\exp\Lambda(-c) + \exp\Lambda(c)\big)$.

About proofs: how to split a Gaussian process...

X is the convolution of the white noise W and a "good" function. Then:



...and how to split its logarithm

Not easy. $\log 0 = -\infty$ What if the Gaussian process is too small on an interval? small ball probability Regretfully, specific properties are used, both of logarithm, and of Gaussian measures.

Random field, dimension 2

Induction in dimension. The gap MDP-LDP becomes $\log^2$.

Splittability involves:

Dimension 1:
three processes X^k for $k = -1, 0, 1$; each distributed like X;
X^-, X^+ are independent;
$X^0(t) \approx X^{\operatorname{sgn} t}(t)$ when $|t| \gg 1$.

Dimension 2:
nine fields $X^{k,l}$ for $k, l = -1, 0, 1$; each distributed like X;
the two triples $(X^{-,-}, X^{-,0}, X^{-,+})$ and $(X^{+,-}, X^{+,0}, X^{+,+})$ are independent;
the two triples $(X^{-,-}, X^{0,-}, X^{+,-})$ and $(X^{-,+}, X^{0,+}, X^{+,+})$ are independent;
$X^{0,0}(s,t) - X^{0,\operatorname{sgn} t}(s,t) - X^{\operatorname{sgn} s,0}(s,t) + X^{\operatorname{sgn} s,\operatorname{sgn} t}(s,t) \approx 0$
whenever $|s| + |t| \gg 1$.

The end