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38th Conference on Stochastic Processes and their Applications

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Overlaps and Pathwise Localization in the Anderson Polymer Model

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SPA, Oxford, 2015

Acknowledgements

- SPA Editorial Board
- Université Diderot - Paris 7
- Kiyoshi Itô

Outline

Polymer Models.

Basic Pinning Model

- A probability measure P on a space of paths, $\omega : [0,\infty) \to Z^d$.
- A functional H_T on paths of length T, e.g. $H_T(\omega) = \int_0^T \delta_0(\omega(s))ds$.
- A new probability measure $d\mu_T^\beta(\omega) = Z_\beta^{-1}(T)\exp\{\beta H_T(\omega)\}dP(\omega)$, $\beta > 0$, on paths $\omega : [0,\infty) \to Z^d$.
- $Z_\beta(T) = E[\exp\{\beta H_T(\omega)\}]$ is called the partition function.

Other examples of polymer models.

Polymer in discrete time and random environment.

- Let $\{S_{i,j} : i \in \mathbf{N}, j \in \mathbf{Z}\}$ be *iid* random variables (Gaussian or Bernoulli, e.g.).
- Let $\omega : \mathbf{N} \to \mathbf{Z}$ be simple symmetric random walk with respect to P.
- Let $H_n(\omega) = \sum_{i=0}^{n} S_{i,\omega(i)}$.
- Set $d\mu_n^\beta(\omega) = Z_\beta^{-1}(n) \exp\{\beta H_n(\omega)\} dP(\omega)$.
- $Z_\beta(n) = E[\exp\{\beta H_n(\omega)\}]$.
- This model was also studied by many authors Imbrie-Spencer, Bolthausen, Carmona-Hu, Comets-Shiga-Yoshida, Rovira-Tindel,....

Anderson polymer model.

Polymer in continuous time and random environment.

- Let $\{B_x(t) : t \in [0,\infty), x \in \mathbf{Z^d}\}$ be *iid* Brownian motions.
- Let $\omega : [0,\infty) \to \mathbf{Z^d}$ be continuous time rate κ simple symmetric random walk with respect to P^κ.
- Let $H_T(\omega) = \int_0^T dB_{\omega(s)}(s)$.
- Set $d\mu_{\kappa,\beta,T}(\omega) = Z^{-1}_{\kappa,\beta}(T)\exp\{\beta H_T(\omega)\}dP^\kappa(\omega)$.
- $Z_{\kappa,\beta}(T) = E^\kappa[\exp\{\beta H_T(\omega)\}]$
- This model was studied by many authors Comets, Carmona-Hu, Vargas,....

Free Energy.

Free Energy

- The partition function can reveal phase transitions (as β changes) in the polymer model.
- This is exhibited by the free energy $F(\beta) \equiv \lim \frac{1}{T} \ln Z_\beta(T)$.
- The existence of this limit is typically established by sub-additivity, such as a sub-additive ergodic theorem, for alternative approach through concentration of measure see Rovira-Tindel.

Free Energy.

Free energy for pinning model.

- For all $\beta > 0$, $F(\beta)$ is the eigenvalue of $\Delta + \beta\delta_0$ if $d = 1$ or 2.
- If $d \geq 3$, there is a $\beta_c > 0$ such that $F(\beta) = 0$ for $\beta \leq \beta_c$ and $F(\beta)$ is the eigenvalue of $\Delta + \beta\delta_0$ if $\beta > \beta_c$.
- The change in behavior of F at $\beta = \beta_c$ signals a phase transition from an extended state to a confined (globular) state for the polymer model.

History

Parabolic Anderson Model

- Let Δ be the discrete Laplacian on $\mathbf{Z^d}$:
  $\Delta f(x) = \frac{1}{2d} \sum_{y:|y-x|=1} (f(y) - f(x))$.
- Original Anderson problem: determine spectrum of $\Delta + \sigma \sum_{x \in \mathbf{Z^d}} \xi_x \delta_x$, where $\sigma > 0$, $\xi_x, x \in \mathbf{Z^d}$ are *iid*.
- Parabolic Anderson Model (PAM), solution of the equation:
  $\frac{\partial u}{\partial t}(t,x) = 1 + \kappa \Delta u(t,x) + \beta u(t,x) \partial B_x(t),\ x \in \mathbf{Z^d}, t \geq 0$.
- In integrated form equation is: $u(t,x) = 1 + \kappa \int_0^t \Delta u(s,x) ds + \beta \int_0^t u(s,x) \partial B_x(s),\ x \in \mathbf{Z^d}, t \geq 0$.
- Feynman-Kac gives representation
  $u(t,x) = E_x^\kappa[\exp\{\beta \int_0^t dB_{\omega(t-s)}(s)\}]$.

Free Energy and Lyapunov Exponent

Free Energy and Lyapunov Exponent

- $\lim_{t\to\infty} \frac{1}{t} \ln u(t,0) = \Psi(\kappa,\beta)$, P^κ *a.s.*. (subadditivity argument C-M-S, 2002)
- Scaling: $\Psi(\kappa,\beta) = \kappa\Psi(1,\kappa^{-1/2}\beta)$.
- Asymptotics: $\Psi(\kappa,1) \sim \frac{\alpha^2}{4\ln\frac{1}{\kappa}}$, $\kappa \searrow 0$. (large deviations)
- $\Psi(\kappa,\beta) = \frac{\beta^2}{2} \iff \frac{\beta^2}{\kappa} \le \Upsilon_c$ for some $\Upsilon_c \in [0,\infty)$
- $Z^\kappa_\beta(T) \stackrel{\mathcal{L}}{=} u(T,0)$,
- $\lim_{t\to\infty} \frac{1}{T} \ln Z^\kappa_\beta(T) = \Psi(\kappa,\beta)$, P^κ *a.s.*.
- This means the free energy for the parabolic Anderson polymer model is the Lyapunov exponent for the PAM.

Overlaps in Statistical Mechanics Models.

Overlaps in Ising Model

- Configuration space $\Sigma = \{-1, 1\}^N$ and Gibbs measure $\mu(\sigma) = Z_N^{-1} \exp\{\sum_{i \neq j} J_{i,j}\sigma_i\sigma_j + h\sum_i \sigma_1\}, J_{i,j} \geq 0$.
- The mean magnetization, $m_N = \frac{1}{N}\sum_i \sigma_i$, is the overlap between σ and the configuration $\hat{\sigma}$ which is given by $\hat{\sigma}_i = 1,\ i = 1, ..., N$.
- That is $m_N = R(\sigma, \hat{\sigma}) \equiv \frac{1}{N}\sum_{i=1,...,N} \sigma_i\hat{\sigma}_i$.

Overlaps in Statistical Mechanics Models.

Overlaps in Sherrington Kirkpatrick

- $H_N(\sigma) = \sum_{i,j=1,\ldots,N, i \neq j} J_{i,j} \sigma_i \sigma_j$ now $J_{i,j}$ are iid $\mathcal{N}(0, \frac{1}{\sqrt{N}})$.
- Gibbs measure $\mu(\sigma) = Z^{-1} \exp\{\beta H_N(\sigma)\}$
- Overlap between two independent selections, σ^1, σ^2 according to distribution μ is given by $R(\sigma_1, \sigma_2) = \frac{1}{N} \sum_{i=1,\ldots,N} \sigma_i^1 \sigma_i^2$.
- This is comparing the model to itself.

Overlaps in the Anderson Polymer Model.

Overlaps in Anderson Polymer Model.

- The path overlap is defined as
  $J_{\kappa,\beta,T} \equiv \frac{1}{T}\int_0^T \mu_{\kappa,\beta,T}^{\otimes 2}(\omega_1(t)=\omega_2(t))dt$
- The endpoint overlap is defined as
  $I_{\kappa,\beta,T} = \frac{1}{T}\int_0^T \mu_{\kappa,\beta,t}^{\otimes 2}(\omega_1(t)=\omega_2(t))dt$.
- Recall $H_T(\omega) = \int_0^T dB_{\omega(s)}(s)$ and
  $d\mu_{\kappa,\beta,T}(\omega) = Z_{\kappa,\beta}^{-1}(T)\exp\{\beta H_T(\omega)\}dP^{\kappa}(\omega)$.

Overlaps and Localization.

Overlaps and Disorder

- Carmona-Hu showed $\int_0^\infty \mu_{\kappa,\beta,t}^{\otimes 2}(\omega_1(t)=\omega_2(t))dt = \infty \iff \lim_{T\to\infty} Z_{\kappa,\beta,T}\exp\{-T\beta^2/2\} = 0,\ a.s.$
- The former is called weak-localization and the latter is called "strong disorder."
- Note $lim_{T\to\infty} Z_{\kappa,\beta,T}\exp\{-T\beta^2/2\} = 0,\ a.s. \iff \Psi(\kappa,\beta) = \frac{\beta^2}{\kappa}$
- Strong localization is the existence of a (random) point x such that $\limsup_{T\to\infty}\sup_x \mu_{\kappa,\beta,T}(\omega(T)=x) > 0$.
- Carmona-Hu, proved in any dimension, strong disorder implies strong localization.

Limit Theorem.

Theorem

- $\tilde{I}_{\kappa,\beta,\infty} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mu^{\otimes 2}_{\kappa,\beta,t}(\omega_1(t)=\omega_2(t))dt$, *a.s.*
- $\tilde{I}_{\kappa,\beta,\infty} = 1 - \frac{2}{\beta^2}\Psi(\kappa,\beta)$.
- $\tilde{J}_{\kappa,\beta,\infty} = \lim_{T\to\infty} E[\frac{1}{T}\int_0^T \mu^{\otimes 2}_{\kappa,\beta,T}(\omega_1(t)=\omega_2(t))dt]$
- $\tilde{J}_{\kappa,\beta,\infty} = 1 - \frac{1}{\beta}\frac{\partial\Psi}{\partial\beta}(\kappa,\beta)$ *except for at most countably any values of* $\frac{\beta^2}{\kappa}$

Corollary

Corollary

- $$\widetilde{I}_{\kappa,\beta,\infty} > 0 \iff \beta^2/\kappa > \Upsilon_c.$$
- *Similarly, for all $\kappa > 0$,*

$$\Upsilon_c = \inf\{\beta^2/\kappa : \widetilde{J}_{\kappa,\beta,\infty} > 0\}.$$

Favorite Endpoint-Favorite Path

Favorite Endpoint-Favorite Path

- $x^*(t) = argmax\{E_\kappa[\exp\{\beta H_t(\omega)\}\delta_x(\omega(t))] \ : x \in \mathbf{Z^d}\}$ (favorite endpoint)
- For fixed t, $x^*(t)$ maximizes $\mu_{\kappa,\beta,t}(\omega(t) = x)$
- $y^*_T(t) = argmax\{E_\kappa[\exp\{\beta H_T(\omega)\}\delta_x(\omega(t))] : \ x \in \mathbf{Z^d}\}$ (favorite path)
- For all $t \in [0, T]$, $y^*_T(t)$ maximizes $\mu_{\kappa,\beta,T}(\omega(t) = x)$

Favorite Endpoint Asymptotic Theorem

Theorem

- *for* $\frac{\beta^2}{\kappa} > \Upsilon_c$, $LI = \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mu_{\kappa,\beta,t}(\omega(t) = x^*(t))dt \geq C > 0$. *(strong disorder localizes endpoint)*
- *for* $\frac{\beta^2}{\kappa} \leq \Upsilon_c$, $LI = 0$. *(weak disorder doesn't localize endpoint)*
- $LS = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mu_{\kappa,\beta,t}(\omega(t) = x^*(t))dt$ *then as* $\frac{\beta^2}{\kappa} \to \infty$,

$$(1-\epsilon)\frac{\alpha^2}{4\ln \beta^2/\kappa} \leq 1 - LS \leq 1 - LI \leq (2+\epsilon)\frac{\alpha^2}{4\ln \beta^2/\kappa}$$

Favorite Path Asymptotic Theorem

Theorem

for all β^2/κ large enough, we have a.s.,

$$\frac{\alpha^2}{8\ln(\beta^2/\kappa)} \leq \liminf_{T\to\infty} E\mu_{\kappa,\beta,T}\left(\frac{1}{T}\int_0^T 1_{\{\omega(t)\neq y_T^*(t)\}}dt\right)$$

$$\limsup_{T\to\infty} E\mu_{\kappa,\beta,T}\left(\frac{1}{T}\int_0^T 1_{\{\omega(t)\neq y_T^*(t)\}}dt\right) \leq \frac{3\alpha^2}{4\ln(\beta^2/\kappa)}$$

Endpoint Overlap

Idea of proof for Endpoint

- Itô's formula gives
  $d \ln Z_{\kappa,\beta,t} =$
  $\beta\mu_{\kappa,\beta,t}(dB_{\omega(t)}(t)) + \frac{\beta^2}{2}\left(1 - \mu^{\otimes 2}_{\kappa,\beta,t}(\omega_1(t) = \omega_2(t))\right) dt$
- $\frac{1}{t} \ln Z_{\kappa,\beta,t} = \frac{1}{t} M_t + \frac{\beta^2}{2}\left(1 - \frac{1}{t} \int_0^t \mu^{\otimes 2}_{\kappa,\beta,s}(\omega_1(s) = \omega_2(s)) ds\right)$
- $d \langle M \rangle_t = \beta^2 \mu_{\kappa,\beta,t}(\omega_1(t) = \omega_2(t)) dt$
- $\lim_{t\to\infty} \frac{1}{t} M_t = 0$

Path Overlap

Idea of proof for Path

- Malliavin derivative: $D_{t,x}H_T(\omega) = \frac{\partial H_T}{\partial dB_x(t)}(\omega) = \delta_x(\omega(t))$.
- Chain rule:

$$\begin{aligned} D_{t,x}Z_{\kappa,\beta,T} &= \beta E_\kappa[\delta_x(\omega(t))\exp\{\beta H_T(\omega)] \\ D_{t,x}\ln Z_{\kappa,\beta,T} &= \beta\mu_{\kappa,\beta,T}(\delta_x(\omega(t))). \end{aligned}$$

- Gaussian IBP $E[H_T(\omega)F] = E[\sum_x \int_0^T \delta_x(\omega(t))D_{t,x}Fdt]$
- $\frac{\partial}{\partial\beta}E[\ln Z_{\kappa,\beta,T}] = \beta E\left[\int_0^T \left(1 - \mu^{\otimes 2}_{\kappa,\beta,T}(\omega_1(t) = \omega_2(t))\right) dt\right]$
- $\lim_{T\to\infty}\frac{1}{T}lnZ_{\kappa,\beta,T} = \lim_{T\to\infty}\frac{1}{T}E[\ln Z_{\kappa,\beta,T}]$.

Favorite Point

Idea of proof for favorite point/path

- $$\begin{aligned} &\frac{1}{T}\int_0^T \mu_{\kappa,\beta,t}^{\otimes 2}(\omega_1(t) \neq \omega_2(t))dt \\ &= \frac{1}{T}\int_0^T \sum_{x\in\mathbb{Z}^d} \mu_{\kappa,\beta,t}(\omega(t) = x)\mu_{\kappa,\beta,t}(\omega(t) \neq x)dt \\ &= \frac{1}{T}\int_0^T \left(1 - \sum_{x\in\mathbb{Z}^d} \mu_{\kappa,\beta,t}(\omega(t) = x)^2\right) dt. \end{aligned}$$
- then split off off the term for $x = x^*(t)$ from the sum and do a little rearrangement
- similar strategy works for favorite path

Jump Distribution

Jump Distribution

- $N(T, \omega)$ is the number of jumps of ω in [0, T]
- For $r \geq 0$, the limit

$$\Gamma(\beta, r) = \lim_{T \to \infty} T^{-1} \ln E_\kappa [\exp\{\beta H_T(\omega)\} | N(T, \omega) = [rT]]$$

  exists a.s. and in L^p, $p \in [1, \infty)$.
- Define $I_\kappa(r) = r \ln(r/\kappa) - r + \kappa$ (Poisson large deviation rate function)

Jump Distribution

Jump Distribution

[Large deviations] Define $\mathrm{I}_{\kappa,\beta}$ to be the convex function

$$I_{\kappa,\beta}(r) = -\Gamma(\beta, r) + \mathrm{I}_{\kappa}(r) + \Psi_{\kappa,\beta}.$$

Then

$$\lim_{T\to\infty, n/T\to r} T^{-1} \ln \mu_{\kappa,\beta,T}\big(N(T,\omega) = n\big) = -I_{\kappa,\beta}(r), \ \ a.s..$$

The usual large deviations hold for $N(T,\omega)$ with respect to $\mu_{\kappa,\beta,T}$ with rate function $I_{\kappa,\beta}$.

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- $$\Gamma_{[0,n],n} \equiv \left\{ \gamma \in \mathcal{D}_n : \gamma : [0,n] \to \mathbb{Z}^d,\ N(\gamma,n) = n \right\}.$$
- $$A_n = \sup_{\gamma \in \Gamma_{[0,n],n}} H_n(\gamma),\ \text{(super-additive)}$$
- $$\lim_{n\to\infty} \frac{1}{n} A_n = \alpha > 0,\ \ a.s.$$

Gaussian Integration by Parts

IBP

$$\begin{aligned}
E\left[\mu_{\kappa,\beta,T}(H_T(\omega))\right] &= E\left[\mu_{\kappa,\beta,T}\left(\sum_{x\in\mathbb{Z}^d}\int_0^T dB_x(t)\delta_x(\omega(t))\right)\right] \\
&= \sum_{x\in\mathbb{Z}^d} EE_\kappa\left[\frac{e^{\beta H_T(X)}}{Z_{\kappa,\beta,T}}\int_0^T dB_x(t)\delta_x(\omega(t))\right] \\
&= \sum_{x\in\mathbb{Z}^d}\int_0^T E_\kappa\left[E\left[\frac{e^{\beta H_T(X)}}{Z_{\kappa,\beta,T}}dB_x(t)\right]\delta_x(\omega(t))\right] \\
&= \sum_{x\in\mathbb{Z}^d}\int_0^T E_\kappa\left[E\left[D_{t,x}\frac{e^{\beta H_T(X)}}{Z_{\kappa,\beta,T}}\right]\delta_x(\omega(t))\right]dt
\end{aligned}$$