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Stochastic Stefan-type Problems and Limit Order Book Models

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TECHNISCHE UNIVERSITÄT DRESDEN

Outline

1. **Modeling Limit Order Books**
   - Limit Order Books
   - An SPDE Model
2. **SPDEs with Moving Boundaries**
   - Noise in ∞ Dim
   - Non-linear Stefan-type SPDEs
   - Existence
3. **Stochastic Chain Rules**

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3. **Stochastic Chain Rules**

An SPDE Model I

Mid price: $p_* := {}^{(p_{\text{bid}} + p_{\text{ask}})}/_{2}$

LOB Model: $v_b(0, p) := \text{LOB}_{buy}(p)$ and $v_s(0, p) := \text{LOB}_{sell}(p)$

Remark: $\mathrm{d}p_*$ depends on $\mathrm{d}v_b$, $\mathrm{d}v_s$ - and vice versa

An SPDE Model I

Mid price: $p_* := (p_{\text{bid}} + p_{\text{ask}})/2$

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Assumption

- infinitesimal tick size $\longrightarrow$ price-time-continuous model,
- log prices $\longrightarrow p \in P = \mathbb{R}$,
- no spread $\longrightarrow p_{\text{bid}} = p_{\text{ask}} = p_*$.

Boundary Conditions

- Type I: Neumann:

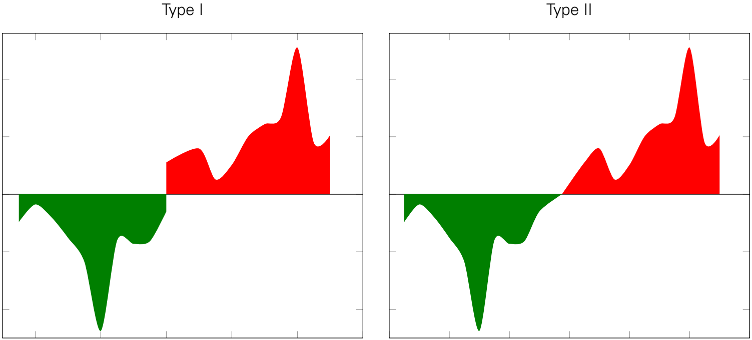
$$\partial_p v(t, p_*\pm) = 0$$

or more general Robin, $\kappa_+, \kappa_- \in (0, \infty)$:

$$v(t, p_*\pm) = \kappa_\pm \partial_p v(t, p_*\pm)$$

- Type II: Dirichlet:

$$v(t, p_*\pm) = 0$$



Price dynamics

- Order-volume at $p_*(t)$: $V_{b/s}(t)$
- Volume imbalance: $\mathcal{I}(t) := \rho(V_b(t) - V_s(t))$, ρ increasing, $\rho(0) = 0$.
- Assumption: $\mathcal{I} > 0 \Rightarrow p_* \nearrow$, $\mathcal{I} < 0 \Rightarrow p_* \searrow$
  (see Cont et al '13, Lipton et al'13)

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Type I: Use best bid/ask queue:

$$V_{b/s}(t) := |v_{b/s}(t, p_*(t)\mp)|$$

Type II: $v_{b/s}(t, p_*(t)\mp) = 0$... Use best bid/ask slope:

$$V_{b/s}(t) := \partial_p v_{b/s}(t, p_*(t)\mp)$$

An SPDE Model II

Initial data: $v_b(0,p) := \mathrm{LOB}_{buy}(p)$ and $v_s(0,p) := \mathrm{LOB}_{sell}(p)$,
ξ: spatially colored noise, B: real BM,

Order Book Dynamics

$$\begin{aligned} \mathrm{d}v_s(t,p) &= \left[\eta_+\partial_p^2 v_s + \mu_+\left(p - p_*(t), v_s, \partial_p v_s\right)\right]\,\mathrm{d}t \\ &\qquad + \sigma_+\left(p - p_*(t), v_s\right)\,\mathrm{d}\xi_t(p), \quad p > p_*(t) \\ \mathrm{d}v_b(t,p) &= \left[\eta_-\partial_p^2 v_b + \mu_-\left(p - p_*(t), v_b, \partial_p v_b\right)\right]\,\mathrm{d}t \\ &\qquad + \sigma_-\left(p - p_*(t), v_b\right)\,\mathrm{d}\xi_t(p), \quad p < p_*(t), \end{aligned}$$

Price Dynamics

$$\begin{aligned} \mathrm{d}p_*(t) = \rho\left(v_b|_{p_*(t)}, v_s|_{p_*(t)}, \partial_p v_b|_{p_*(t)-}, \partial_p v_s|_{p_*(t)+}\right)\,\mathrm{d}t \\ + \sigma_*\,\mathrm{d}B_t. \end{aligned}$$

Outline

Noise I

U, U_1 separable Hilbert spaces s. t. $U \hookrightarrow U_1$ and the embedding is Hilbert-Schmidt.

Definition

(e_k) CONS of U and β_k, $k \in \mathbb{N}$ be ind. Brownian motions. Define the **cylindrical Wiener process** W

$$W_t := \sum_{k=1}^{\infty} e_k \beta_k(t), \ \ t \geq 0,$$

where the series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U_1)$.

Noise II

Example

For $U := L^2(\mathbb{R})$, $(\alpha_k) \in \ell^2$, $\alpha_k > 0$, define $Je_k := \alpha_k e_k$, $U_1 := JU$, and

$$W_t := \sum_k e_k \beta_k(t) := \sum_{\boldsymbol{k}} \boldsymbol{\alpha_k e_k \beta_k(t)}, \quad t \geq 0.$$

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Example

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Definition (Colored noise)

For some $\zeta \in C^\infty(\mathbb{R}) \cap H^3(\mathbb{R})$

$$\xi_t(p) := \int_0^t \zeta * \mathrm{d}W_s(p), \quad t \geq 0,\ p \in \mathbb{R}.$$

Note: $(\zeta * .)(p) \in \mathrm{HS}(U; \mathbb{R})$, for $p \in \mathbb{R}$.

Centered equations

Formally, with $u(t,x) := v(t, x + p_*(t))$ we get the SPDE on $\mathbb{R} \setminus \{0\}$,

(cMBP)

$$
\begin{aligned}
\mathrm{d}u_t(x) &= \left[(\eta + \tfrac{1}{2}\sigma_*^2)\Delta u_t + \mu\,(x, u_t, \nabla u_t) + \mathcal{I}(u_t)\nabla u_t\right]\,\mathrm{d}t \\
&\qquad + \sigma\,(x, u_t)\,\mathrm{d}\xi_t(x + p_*(t)) + \sigma_* \nabla u_t\,\mathrm{d}B_t, \qquad x \neq 0, \\
\mathrm{d}p_*(t) &= \mathcal{I}(u_t)\,\mathrm{d}t + \sigma_*\,\mathrm{d}B_t, \\
u_t(0+) &= \kappa \nabla u_t(0+), \\
u_t(0-) &= -\kappa \nabla u_t(0-),
\end{aligned}
$$

where $\eta > 0$, $\sigma_* \geq 0$, $\kappa \in [0, \infty]$, $\mu : \mathbb{R}^3 \to \mathbb{R}$, $\sigma : \mathbb{R}^2 \to \mathbb{R}$, and

$$\mathcal{I}(u) := \rho(u(0+), u(0-), \nabla u(0+), \nabla u(0-)).$$

Transformation

Set

- $\mathcal{L}^2 := L^2(\mathbb{R}) \oplus \mathbb{R}$, $\mathcal{H}^k := H^k(\dot{\mathbb{R}}_0) \oplus \mathbb{R}$, where $\dot{\mathbb{R}}_0 := \mathbb{R} \setminus \{0\}$
- $\bar{W} := (W_t, B_t)$ is a cylindrical Wiener process on $U := L^2(\mathbb{R}) \oplus \mathbb{R}$,

The problem (cMBP) can be written as

$$\text{(EE)} \qquad \mathrm{d}X(t) = [AX(t) + B(X(t))] \,\mathrm{d}t + C(X(t)) \,\mathrm{d}\bar{W}_t,$$

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For Δ Laplacian and ∇ gradient on $\mathbb{R} \setminus \{0\}$,

$$A = \begin{pmatrix} (\eta + \frac{1}{2}\sigma_*^2)\Delta & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{D}(A) = \{(u, x) \in \mathcal{H}^2 \mid \nabla u(0+) = \kappa u(0+),\ \nabla u(0-) = -\kappa \nabla u(0-)\}$$

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(EE) $$\mathrm{d}X(t) = [AX(t) + B(X(t))]\,\mathrm{d}t + C(X(t))\,\mathrm{d}\bar{W}_t,$$

For $u \in \mathcal{L}^2$, $x \in \mathbb{R}$:

$$B(u)(x) = \begin{pmatrix} \mu(x, u_1(x), \nabla u_1(x)) + \nabla u_1(x) \cdot \mathcal{I}(u) \\ \mathcal{I}(u) \end{pmatrix},$$

$$\mathcal{I} : u \mapsto \rho(u_1(0+), u_1(0-), \nabla u_1(0+), \nabla u_1(0-)),$$

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(EE) $$\mathrm{d}X(t) = [AX(t) + B(X(t))]\,\mathrm{d}t + C(X(t))\,\mathrm{d}\bar{W}_t,$$

For $u \in \mathcal{L}^2$, $w \in L^2(\mathbb{R})$, $x, b \in \mathbb{R}$:

$$C(u)(w,b)(x) = \begin{pmatrix} \sigma(x, u_1(x)) \cdot (\zeta * w)(u_2 + x) \\ 0 \end{pmatrix} + \sigma_* b \begin{pmatrix} \nabla u(x) \\ 1 \end{pmatrix}.$$

The Spaces

Remark (Fractional order domains)

For $\alpha \in \mathbb{R}$ define the Hilbert spaces

$$E_\alpha := \mathcal{D}\left((a - A)^\alpha\right), \qquad \|u\|_\alpha := \|(a - A)^\alpha u\|, \qquad a > 0.$$

Then, $E_0 = \mathcal{L}^2$ and

$$E_1 \approx \mathcal{D}(A) \subset \mathcal{H}^2 \hookrightarrow BUC^1(\dot{\mathbb{R}}_0) \oplus \mathbb{R}.$$

Note that $H^k(\mathbb{R} \setminus \{0\})$ can be identified with $H^k(\mathbb{R}_+) \oplus H^k(\mathbb{R}_+)$,

The Coefficients I

Remark

For all $a > 0$, $a - A$ is positive self-adjoint on $\mathcal{L}^2$, so that

- A generates the analytic contraction semigroup $(S_t) := (e^{tA})$,
- $S_t u \in \cap_{n\in\mathbb{N}} \mathcal{D}(A^n)$, for all $u \in \mathcal{L}^2$, $t > 0$.
- For all $\alpha \in [0, 1]$ and $t > 0$, $u \in E_\alpha$,

$$\|S_t u\|_1 \lesssim t^{\alpha-1} \|u\|_\alpha .$$

Recall the variation of constants formula for (EE),

$$\mathrm{d}X_t = S_t X_0 + \int_0^t S_{t-s} B(X(s))\, \mathrm{d}s + \int_0^t S_{t-s} C(X(s))\, \mathrm{d}\bar{W}_s$$

The Coefficients II

Lemma

If $\rho : \mathbb{R}^4 \to \mathbb{R}$ is locally Lipschitz, then $\mathcal{I}$ and

$$H^2(\dot{\mathbb{R}}_0) \ni \; u \mapsto \mathcal{I}(u)\partial_x u \; \in H^1(\dot{\mathbb{R}}_0)$$

are Lipschitz on bounded sets.

Hence, $B : \mathcal{H}^2 \to \mathcal{H}^1$ is Lipschitz on bd. sets, if

- $\mu(x, y, z) = \mu_+(x, y, z)\mathbf{1}_{\mathbb{R}_+}(x) + \mu_-(x, y, z)\mathbf{1}_{\mathbb{R}_-}(x)$, $x, y, z \in \mathbb{R}$,
- $\mu_+, \mu_- \in C^1(\mathbb{R}^3)$, s. t.

  $$\left|\partial_x^{(i)} \mu_\pm(x, y, z)\right| \leq a(x) + b(y, z)\,|(y, z)|$$

  and

  $$\sup_{x\in\mathbb{R}} |\partial_y \mu_\pm(x, y, z)| + \sup_{x\in\mathbb{R}} |\partial_z \mu_\pm(x, y, z)| \leq b(y, z)$$

  for $a \in L^2(\mathbb{R})$ and $b \in L^\infty_{loc}$.
- ... related Lipschitz conditions

The Coefficients III

Write $C(u)[w, b] = C_1(u)w + C_2(u)b$. Assume

- $\sigma \in C^2(\mathbb{R}^2)$, and GC and LC, similar to μ
- boundary conditions for σ, e. g. for Dirichlet $\sigma(0,0) = 0$ or for Robin,

$$\sigma(x, y) = \partial_x \sigma(x, y) = \partial_y \sigma(x, y) = 0, \ \forall x, \ y \in \mathbb{R}.$$

- if $\sigma_* > 0$, then assume $\kappa \in (0, \infty]$ (Neumann or Robin BC).

Then, $C_1 : \mathcal{D}(A) \to HS(L^2(\mathbb{R}); \mathcal{D}(A))$ is Lipschitz on bd sets.

Remark

$$E_1 = \mathcal{D}(A) \subset \mathcal{H}^2, \quad \text{and} \quad \mathcal{H}^{2\alpha} \hookrightarrow E_\alpha, \forall \alpha < 1/4.$$

Moreover, for the Neumann and Robin BC holds

$$\mathcal{H}^1 \hookrightarrow E_{1/2},$$

for Dirichlet not!

Stoch Max L^p Regularity

Assumptions

- S_t is "sufficiently regular"
- $B : E_1 \to E_\alpha$, $\alpha > 0$, Lipschitz on bd. sets
- $C = C_1 + C_2$, $C_1 : E_1 \to HS(U; E_1)$ Lipschitz on bd. sets
- $C_2 : E_1 \to HS(U; E_{1/2})$ Lipschitz with $\|C_2\|_{LIP} \leq 1/\sqrt{2}$.

Theorem (van Neerven, Veraar, Weis (2012))

The equation

$$\mathrm{d}X(t) = [AX(t) + B(X(t))]\,\mathrm{d}t + C(X(t))\,\mathrm{d}W_t$$

with initial data $X(0) = X_0 \in L^2(\Omega; E_1)$ admits an unique strong maximal solution (X, τ) in

$$L^0(\Omega; L^2(0, \tau; E_1)) \cap L^0(\Omega; C([0, \tau); E_{1/2})).$$

Stoch Max L^p Regularity

Assumptions

- S_t is "sufficiently regular" ok
- $B : E_1 \to E_\alpha$, $\alpha > 0$, Lipschitz on bd. sets ok
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- $C_2 : E_1 \to HS(U; E_{1/2})$ Lipschitz with $\|C_2\|_{LIP} \leq 1/\sqrt{2}$. $\to$ $\sigma_* < K/\sqrt{2}$

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Outline

Stochastic Chain Rules

Recall $(u_1, u_2, p_*) = X$ and for $x \in \mathbb{R}$, $t \geq 0$,

$$v(t) = \Theta(u(t), p_*(t)) := u(t, (.) - p_*(t))$$

Change of coordinates was only formal!

Question

Relationship between the centered equations and the model on the randomly moving frame?

On Ito's Formula

In general, $(u, x) \in \mathcal{H}^1$ holds $\Theta(u, x) \notin H^1(\mathbb{R})$!

- $\Theta \notin C^1(\mathcal{H}^1; \mathcal{L}^2)$, and, in particular,
- $\Theta \notin C^2(\mathcal{H}^2; \mathcal{L}^2)$,

$\Rightarrow$ No chance for Ito formula in this setting!

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Natural Idea:

Adjoint of $\Theta(., x)$ is $\Theta(., -x)$!

1. Test v against $\phi \in C_0^\infty(\mathbb{R})$
2. Put the shift $x \mapsto x - p_*(t)$ into the test function

This is the idea of the so called **Ito-Wentzell formula** for SPDEs (Krylov 1999, 2010)

Ito-Wentzell Formula

u is a solution of

$$du(t,x) = ((\eta + \tfrac{1}{2}\sigma_*^2)\Delta u(t,x) + \nabla u(t,x)\mathcal{I}(u(t)))\,\mathrm{d}t + \mu(x, u(t,x), \nabla u(t,x))\,\mathrm{d}t + \sigma(x, u(t,x))\,\mathrm{d}\xi_t(x + p_*(t)) + \sigma_* \nabla u(t,x)\,\mathrm{d}\beta_t,$$

in distributional sense, iff $v(t,x) = u(t, x + p_*(t))$ is solves

$$\mathrm{d}v(t,x) = \eta\Delta v(t,x) + \mu(x - p_*(t), v(t,x), \nabla v(t,x),)\,\mathrm{d}t + \sigma(x, v(t,x))\,\mathrm{d}\xi_t(x) + L_1(v(t), p_*(t))\,\mathrm{d}p_*(t) + L_2(v(t), p_*(t))\,\mathrm{d}\langle p_*\rangle_t,$$

where L_1 and L_2 are (given) distribution valued functions with

$$L_1(.,p)|_{\mathbb{R}\setminus\{p\}} = L_2(.,p)|_{\mathbb{R}\setminus\{p\}} = 0.$$

Ito-Wentzell Formula II

Warning

∇ and Δ are piecewise (!) weak derivatives and not the distributional derivatives on $\mathbb{R}$!
E. g. for $f \in H^1(\mathbb{R} \setminus \{0\})$,

$$\partial_x f - \nabla f = (f(0+) - f(0-))\delta_0.$$

Remark

At least formally, L_1 and L_2 describe the jumps of v and ∇v at phase change p_*.

Summary and Outlook

- Strong local solution of the stochastic evolution equation,
- If $\sigma_* > 0$, then only Neumann and Robin bc.
- Transformation in general via Ito-Wentzell formula - in distributional sense.

Special case $\sigma_* = 0$

- All (hom.) boundary conditions possible,
- Continuity w.r.t. $\mathcal{H}^2$,
- Analysis much easier,
- Forward invariance (positivity),
- Wong-Zakai approximation,
- For Dirichlet b. c. : $\Theta \in C^1(\mathcal{H}^1_0; L^2(\mathbb{R}))$ yields strong solution of the SMBP. ($\rightarrow$ arXiv 1507.03276)

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