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Random walk loop percolation

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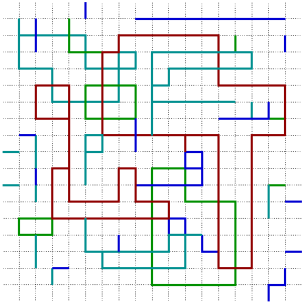
Markovian loop percolation: rough description

- $\mathbb{Z}^d$, $d \geq 3$
- random walk loop soup
- loop clusters and loop percolation



Loop soup

- Discrete based loops $\dot{\ell} = (x_1, \ldots, x_n)$: nearest neighbor walk $x_1 \to x_2 \to \cdots \to x_n \to x_1$
- Measure on discrete based loops:

$$\dot{\mu}((x_1, \ldots, x_n)) = \frac{1}{n} \left(\frac{1}{2d}\right)^n.$$

- Discrete loop $\ell = [\dot{\ell}]$, equivalence class of $\dot{\ell}$
- Loop measure μ, push-forward of $\dot{\mu}$
- PPP $\mathcal{L}_\alpha$ on discrete loops with intensity $\alpha\mu$

$$\mathbb{P}[\ell \in \mathcal{L}_\alpha] = 1 - e^{-\alpha\mu(\ell)}$$

Symanzik, Lawler, Le Jan, Sznitman

Loop percolation

- Edge e open $\iff \exists \ell \in \mathcal{L}_\alpha$ traversing e
- Bernoulli bond percolation = percolation restricted on loops with length 2
- Percolation probability: $\theta(\alpha) \stackrel{\text{def}}{=} \mathbb{P}_\alpha[0 \longleftrightarrow \infty]$
- Phase transition:
  - $\alpha_c(d) \stackrel{\text{def}}{=} \inf\{\alpha > 0 : \theta(\alpha) > 0\}$
    For $d \geq 3$, $\frac{1}{2} \leq \alpha_c(d) < \infty$, $\alpha_c(d) = 2d + O(1)$ (Y. Le Jan and S. Lemaire; T. Lupu; A. Sapozhnikov and C.)
  - $\alpha < \alpha_c$: all clusters are finite almost surely
  - $\alpha > \alpha_c$: $\exists$ unique infinite cluster almost surely

Goals

- Geometric properties of finite clusters in sub/supercritical regime
- Geometric properties of the unique infinite cluster, e.g. simple random walk on the infinite cluster

Difficulties

- Long range correlation: for $d \geq 3$, edges e and f,

$$\mathbb{P}[e, f \text{ open}] - \mathbb{P}[e \text{ open}]\mathbb{P}[f \text{ open}] \asymp (1 + dist(e, f))^{2(2-d)}.$$

- Other models with long range correlations:
  - Vacant set of random interlacement
  - Level set of gaussian free field
- Power law decay of connectivity: for $d \geq 3$,

$$\mathbb{P}[\exists \ell \in \mathcal{L}_\alpha : 0, x \in \ell] \asymp \alpha(1 + |x|)^{2(2-d)},$$

$$\mathbb{P}[\exists \ell \in \mathcal{L}_\alpha : 0 \in \ell, \ell \cap \partial B(n)] \asymp \alpha n^{2-d}.$$

In particular, $\mathbb{P}_\alpha[0 \longleftrightarrow \partial B(n)] \geq c(d, \alpha) n^{2-d}$ for $\alpha > 0$.

- Lack of decoupling inequality

Subcritical regime, $d \geq 5$

$$\alpha_\# \stackrel{\text{def}}{=} \sup\{\alpha > 0 : \mathbb{E}_\alpha[\#\mathcal{C}(0)] < \infty\} \leq \alpha_c.$$

Theorem (Sapozhnikov and C.)

$\alpha_\# > 0$ when $d \geq 5$ and $\alpha_\# = 0$ when $d = 3, 4$.
For $\alpha < \alpha_\#$,

- $\mathbb{P}_\alpha[0 \longleftrightarrow \partial B(n)] \asymp n^{2-d}$,
- $\mathbb{P}_\alpha[0 \longleftrightarrow x] \asymp (1 + |x|)^{2(2-d)}$,
- $\mathbb{P}_\alpha[\#\mathcal{C}(0) > n] \asymp n^{1-d/2}$.

- For $d \geq 5$, the typical scenario is one big loop near 0.
- Conjecture: $\alpha_c = \alpha_\#$.

Subcritical regime, $d = 3, 4$

$$\alpha_1 \stackrel{\text{def}}{=} \sup\{\alpha > 0 : \liminf_n \mathbb{P}_\alpha[B(n) \leftrightarrow \partial B(2n)] < 1\} \le \alpha_c.$$

Theorem (Sapozhnikov and C.)

For $d \ge 3$, $\alpha_1 > 0$.
For $\alpha < \alpha_1$,

$$\mathbb{P}_\alpha[0 \longleftrightarrow \partial B(n)] \le C(d,\alpha) n^{-c(d,\alpha)}.$$

For $\alpha > 0$,

$$\mathbb{P}_\alpha[0 \longleftrightarrow \partial B(n)] \ge \begin{cases} c_3(\alpha) n^{-1+\epsilon_3(\alpha)}, & \text{for } d = 3, \\ c_4(\alpha) n^{-2} (\log n)^{\epsilon_4(\alpha)}, & \text{for } d = 4. \end{cases}$$

- $\epsilon_3(\alpha) > 0$ and $c(d,\alpha) \to d - 2$ as $\alpha \to 0$
- Big cluster appears because of chain of many loops
- Conjecture: $\alpha_c = \alpha_1$

Supercritical regime

Theorem (C.)

$\forall \alpha > \alpha_c$:

- *Exponential decay of big finite cluster:*

$$\mathbb{P}_\alpha[\#\mathcal{C}(0) < \infty, \mathcal{C}(0) \cap \partial B(n) \neq \emptyset] \leq C(d,\alpha)e^{-c(d,\alpha)n}.$$

- *Unique infinite cluster*
    - *Quenched gaussian type heat kernel estimates*
    - *Quenched invariance principle and local CLT*
    - *Harnack ineq. for large balls*

- RW on ∞ cluster in supercritical Bernoulli percolation:

  Barlow, Mathieu-Remy, Sidoravicius-Sznitman, Berger-Biskup, Mathieu-Piatnitski, Barlow-Hambly

- RW on ∞ cluster in random interlacement ($\forall u > 0$), vacant set of random interlacement and level sets of GFF:

  Procaccia-Rosenthal-Sapozhnikov, Sapozhnikov

Idea of the proof

- Truncated loop soup $\mathcal{L}_\alpha^{\leq L}$, like Bernoulli perc. $\alpha_c^{\leq L} \downarrow \alpha_c$
- Techniques developed for random interlacement + perturbation
- $\mathcal{C}_{\max}(B(n))$: largest cluster (in volume) inside $B(n)$
- $\mathcal{C}_{\max}^{\leq L}(B(n))$: largest cluster of truncated loop percolation
- for $d \geq 3$, $\alpha > \alpha_c$ and $\epsilon > 0$, $\exists \widetilde{L} = \widetilde{L}(d, \alpha, \epsilon)$, $c = c(d, \alpha, \epsilon) > 0$ and $C = C(d, \alpha, \epsilon) < \infty$ s.t. $\forall L \geq \widetilde{L}$,

$$\mathbb{P}_\alpha \left[\#\mathcal{C}_{\max}(B(n)) - \#\mathcal{C}_{\max}^{\leq L}(B(n)) > \epsilon n^d \right] < C e^{-cn^{\frac{d}{d+1}}}.$$

Some references

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