

Viscosity Solutions of Fully Nonlinear Path-Dependent PDEs

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Outline

- 1 Motivation and examples
 - Parabolic nonlinear PPDEs
 - Differentiability of processes
 - Examples
- 2 Definition of viscosity solutions
 - M.Crandall & PL.Lions definition
 - Viscosity solutions of path-PDEs
- 3 Main results
 - Consistency, stability, wellposedness
 - Comparison for semilinear path-dependent PDEs
 - Comparison for fully-nonlinear path-dependent PDEs

Paths space and non-anticipative process

- $\Omega = \{\omega \in C^0([0, T], \mathbb{R}^d), \omega_0 = 0\}$, $\|\omega\| = \sup_{t \leq T} |\omega_t|$
- B canonical process, i.e. $B_t(\omega) = \omega(t)$
- $\mathbb{F} = \{\mathcal{F}_t\}$ the corresponding filtration, i.e. $\mathcal{F}_t = \sigma(B_s, s \leq t)$
- $\Lambda = [0, T] \times \Omega$, $d[(t, \omega), (t', \omega')] = |t - t'| + \|\omega_{\cdot \wedge t} - \omega'_{\cdot \wedge t'}\|$
- $u : \Lambda \rightarrow \mathbb{R}$ **non-anticipative** if $u(t, \omega) = u(t, (\omega_s)_{s \leq t})$

In particular, $u \in C^0(\Lambda) \implies u$ non-anticipative

Probability measures on the paths space

- \mathbb{P}_0 : Wiener measure on Ω , so that B is a \mathbb{P}_0 -Brownian motion
- \mathcal{P}_L : collection of all $\mathbb{P} = \mathbb{P}^{\alpha, \beta}$ such that

$$B_t = \int_0^t \alpha_s^{\mathbb{P}} ds + \int_0^t \beta_s^{\mathbb{P}} dW_t^{\mathbb{P}}, \quad \mathbb{P} - \text{a.s.} \quad \text{for some}$$

- adapted processes $\alpha^{\mathbb{P}}, \beta^{\mathbb{P}}$, with $|\alpha^{\mathbb{P}}| \leq L$ and $\frac{1}{2}|\beta^{\mathbb{P}}|^2 \leq L$
- and \mathbb{P} -Brownian motion $W^{\mathbb{P}}$

In particular,

- $\mathcal{P}_0 = \{\mathbb{P}_0\}$
- Quadratic variation : $\langle B \rangle_t = \int_0^t (\beta_s^{\mathbb{P}})^2 ds, \mathbb{P}^{\alpha, \beta} - \text{a.s.}$

Our objective : nonlinear path-dependent PDEs

Find **non-anticipative** process $u(t, \omega)$ satisfying :

$$\begin{aligned}
 -\partial_t u - G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u) &= 0, \quad \text{on } [0, T) \times \Omega, \\
 u(T, \omega) &= \xi(\omega)
 \end{aligned}$$

where $\xi(\omega) = \xi((\omega_s)_{s \leq T})$ and $G(t, \omega, y, z, \gamma)$ is non-anticipative

$$G : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$$

$$G(t, \omega, y, z, \gamma) = G(t, (\omega_s)_{s \leq t}, y, z, \gamma)$$

Differentiability of processes

Definition

$\varphi \in C^{1,2}(\Lambda)$ if $\varphi \in C^0(\Lambda, \mathbb{R})$ and

there exist $\theta \in C^0(\Lambda, \mathbb{R})$, $Z \in C^0(\Lambda, \mathbb{R}^d)$, $\Gamma \in C^0(\Lambda, \mathbb{S}^d)$ s.t.

$$d\varphi_t = \theta_t dt + Z_t dB_t + \frac{1}{2} \Gamma_t d\langle B \rangle_t, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \cup_{L>0} \mathcal{P}_L$$

Denote $\partial_t \varphi := \theta$, $\partial_\omega \varphi := Z$, and $\partial_{\omega\omega}^2 \varphi := \Gamma$

- In particular, consistent with Dupire's definition of the right time-derivative :

$$\partial_t \varphi(t, \omega) := \lim_{h \rightarrow 0^+} \frac{1}{h} \left[\varphi(t+h, \omega_{\cdot \wedge t}) - \varphi(t, \omega) \right], \quad \text{if exists}$$

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Example 1 : Smooth backward SDEs

- Backward SDE (Pardoux & Peng '91...) :

$$dY_t = -F_t(\omega, Y_t, Z_t)dt + Z_t dB_t, \quad Y_T = \xi, \quad \mathbb{P}_0 - \text{a.s.}$$

if $u(t, \omega) := Y_t(\omega)$ is $C^{1,2}$, i.e.

$$du_t = \left(\partial_t u_t + \frac{1}{2} \partial_{\omega\omega}^2 u_t \right) dt + \partial_{\omega} u_t dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

Then, u solves the **semilinear P-PDE**

$$-\partial_t u - \frac{1}{2} \partial_{\omega\omega}^2 u - F(\cdot, u, \partial_{\omega} u) = 0, \quad u_T = \xi$$

Note : Existing literature establishes wellposedness of the backward SDE in the space

$$\mathbb{E}^{\mathbb{P}_0} \left[\int_0^T (|Y_t|^2 + |Z_t|^2) dt \right] < \infty$$

i.e. Sobolev solutions of P-PDE (Barles & Lesigne '94)



Example 2 : Smooth second order backward SDEs

Cheridito, Soner, NT & Victoir '06
Soner, NT & Zhang '10

... Path-dependent fully nonlinear PDE

including path-dependent HJB equation...

Example 3 : Stochastic control of path-dependent diffusions

- Stochastic control of non-Markov systems :

$$dX_t^\alpha = b(t, X^\alpha, \alpha_t) dt + \sigma(t, X^\alpha, \alpha_t) dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

for some $b : \Lambda \times A \rightarrow \mathbb{R}^d$, $\sigma : \Lambda \times A \rightarrow \mathbb{S}^d$, and

$$u(t, x) := \inf_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[\int_t^T L(s, X^\alpha, \alpha_s) ds + \xi((X^\alpha_s)_{s \leq T}) \right]$$

\implies Path-dependent HJB equation

$$-\partial_t u - \inf_{a \in A} \left\{ b(\cdot, a) \partial_\omega u + \frac{1}{2} \sigma^2(\cdot, a) \partial_{\omega\omega}^2 u + L(\cdot, a) \right\} = 0, \quad u_T = \xi$$

- Alternative approach to [control of hereditary systems...](#), see Cosso, Federico, Gozzi, Rosestolato & NT '14

OBJECTIVE

Inspired by fully nonlinear PDEs in finite dimensional spaces,
we want to develop a theory of viscosity solutions for
path-dependent parabolic fully nonlinear equations

- Existence
- Uniqueness implied by a comparison result (maximum principle)
- Powerful stability result

Applications

Numerical implications :

- Extension of Barles-Souganidis Monotone schemes (Zhang & Zhuo '13 and Ren & Tan '15)
- Branching diffusion representation for path-dependent KPP equations \implies Monte Carlo approximation (Henry-Labordère, Tan & NT)

Regular and singular perturbation (Ma, Ren, Zhang & NT)

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Standard viscosity solutions [M. Crandall & P.-L. Lions '83]

$g(x, y, z, \gamma)$ nondecreasing in γ . Consider the PDE :

$$(E) \quad -g(\cdot, v, Dv, D^2v)(x) = 0, \quad x \in \mathcal{O} \quad (\text{open subset of } \mathbb{R}^d)$$

- v **sub**solution if $-g(\cdot, v, Dv, D^2v) \leq 0$ on \mathcal{O}
- v **super**solution if $-g(\cdot, v, Dv, D^2v) \geq 0$ on \mathcal{O}

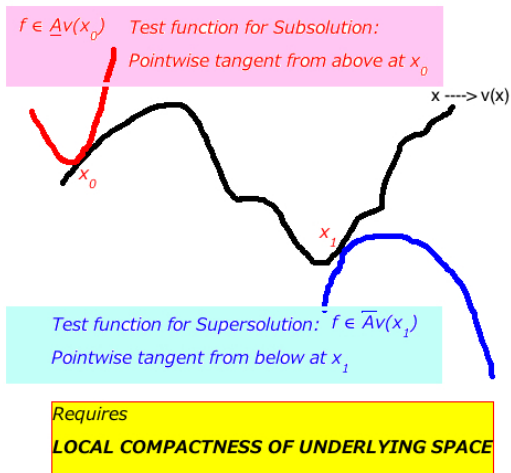


Figure: M.Crandall & PL.Lions test functions

Standard definition of viscosity solutions

Let

$$\underline{Av}(x) := \{\varphi \in C^2(\mathcal{O}) : (\varphi - v)(x) = \min_{\mathcal{O}}(\varphi - v)\}$$

$$\overline{Av}(x) := \{\varphi \in C^2(\mathcal{O}) : (\varphi - v)(x) = \max_{\mathcal{O}}(\varphi - v)\}$$

Definition $v \in LSC(\mathcal{O})$ (resp. $USC(\mathcal{O})$) is a viscosity **sub**solution (resp. **super**solution) of (E) if :

$$-g(x, v(x), D\varphi(x), D^2\varphi(x)) \leq 0 \text{ (resp. } \geq 0)$$

$$\text{for all } x \in \mathcal{O} \text{ and } \varphi \in \underline{Av}(x) \text{ (resp. } \overline{Av}(x))$$

$v \in C^0(\mathcal{O})$ viscosity solution if viscosity subsol. and supersol.

Main properties of Crandall-Lions viscosity solutions

- Stability result : "five lines" proof
- Existence for value function of control problems (representation) : usually very easy
- Uniqueness : implied by comparison result. **This is the difficult part of the theory in the second order case...** (Crandall-Ishii Lemma)
- Existence by Perron method, given comparison
- Regularity : Caffarelli & Cabre

Intuition from the heat equation

$v(t, x) := \mathbb{E}^{\mathbb{P}^0} [g(B_T) | B_t = x]$ solution of $-\partial_t v - \frac{1}{2} v_{xx} = 0$.

- Tower property :

$$\begin{aligned} v(t, x) &= \mathbb{E}^{\mathbb{P}^0} [\mathbb{E}^{\mathbb{P}^0} [g(B_T) | B_{t+h}] | B_t = x] \\ &= \mathbb{E}^{\mathbb{P}^0} [v(t+h, B_{t+h}) | B_t = x] \end{aligned}$$

- Viscosity subsol : for $\varphi \in \underline{A}v(t, x)$, $v(t, x) = \varphi(t, x)$ and $v \leq \varphi$

$$\begin{aligned} \implies \varphi(t, x) &\leq \mathbb{E}^{\mathbb{P}^0} [\varphi(t+h, B_{t+h}) | B_t = x] \\ \implies \left(-\partial_t \varphi - \frac{1}{2} \varphi_{xx} \right)(t, x) &\leq 0 \end{aligned}$$

Main observation : only need $\mathbb{E}^{\mathbb{P}^0} [v(., .)] \leq \mathbb{E}^{\mathbb{P}^0} [\varphi(., .)]$



Smooth test processes

Nonlinear expectation $\bar{\mathcal{E}}_L := \sup_{\mathbb{P} \in \mathcal{P}_L} \mathbb{E}^{\mathbb{P}}$ and $\underline{\mathcal{E}}_L := \inf_{\mathbb{P} \in \mathcal{P}_L} \mathbb{E}^{\mathbb{P}}$

\mathcal{T} : collection of all **stopping times** τ (i.e. $\{\tau \leq t\} \in \mathcal{F}_t$)

Test processes for **sub**solution and **super**solution

$$\underline{\mathcal{A}}^t u_t(\omega) := \left\{ \varphi \in C^{1,2}(\Lambda) : (\varphi - u^{t,\omega})_0 = \min_{\tau \in \mathcal{T}} \underline{\mathcal{E}}_L [(\varphi - u^{t,\omega})_{\cdot, \Lambda_H}] \right. \\ \left. \text{for some } H \in \mathcal{H} \right\}$$

$$\overline{\mathcal{A}}^t u_t(\omega) := \left\{ \varphi \in C^{1,2}(\Lambda) : (\varphi - u^{t,\omega})_0 = \max_{\tau \in \mathcal{T}} \bar{\mathcal{E}}_L [(\varphi - u^{t,\omega})_{\cdot, \Lambda_H}] \right. \\ \left. \text{for some } H \in \mathcal{H} \right\}$$

TEST FUNCTIONS TANGENT IN MEAN

Definition (Ekren, NT & Zhang 2012)

$u \in C^0(\Lambda)$ is a **viscosity**...

- **subsolution** of PPDE if there exists L :

$$-\partial_t \varphi_0 - G(t, \omega, u_t(\omega), \partial_\omega \varphi_0, \partial_{\omega\omega}^2 \varphi_0) \leq 0$$

for all $(t, \omega) \in [0, T) \times \Omega$ and $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$

- **supersolution** of PPDE if there exists L :

$$-\partial_t \varphi_0 - G(t, \omega, u_t(\omega), \partial_\omega \varphi_0, \partial_{\omega\omega}^2 \varphi_0) \geq 0$$

for all $(t, \omega) \in [0, T) \times \Omega$ and $\varphi \in \overline{\mathcal{A}}^L u(t, \omega)$

- **solution** of PPDE if it is viscosity **subsolution** and **supersolution**

Equivalent semijets definition

For $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^d$, $\gamma \in \mathbb{S}_d$, denote the **paraboloid** :

$$Q^{\alpha, \beta, \gamma}(t, \omega) := \alpha t + \beta \cdot \omega_t + \frac{1}{2} \gamma \omega_t \cdot \omega_t$$

- **subset** : $\underline{\mathcal{J}}^L u_t(\omega) := \{(\alpha, \beta, \gamma) : Q^{\alpha, \beta, \gamma} \in \underline{\mathcal{A}}^L u_t(\omega)\}$,
- **superjet** : $\overline{\mathcal{J}}^L u_t(\omega) := \{(\alpha, \beta, \gamma) : Q^{\alpha, \beta, \gamma} \in \overline{\mathcal{A}}^L u_t(\omega)\}$

Theorem

Viscosity subsolution and supersolution can be reduced to paraboloids

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Consistency with classical solutions

- Assumption G1** $G(t, \omega, y, z, \gamma)$ nondecreasing in γ and satisfies :
- (i) G is uniformly continuous in (t, ω) , and $\|G(\cdot, 0, 0, 0)\|_\infty < \infty$.
 - (ii) G is uniformly Lipschitz in (y, z, γ)

Theorem (Ekren, NT & Zhang 2012a)

Let Assumption G1 hold and $u \in C_b^{1,2}(\Lambda)$. Then the following assertions are equivalent :

- u classical solution (resp. subsolution, supersolution) of PPDE
- u viscosity solution (resp. subsolution, supersolution) of PPDE

Stability

Consider the perturbed PPDEs

$$\text{PPDE}^\varepsilon : G^\varepsilon(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u) = 0 \text{ on } [0, T] \times \Omega$$

Theorem (Ekren, NT & Zhang 2012a)

Let u^ε viscosity L -subsolution (resp. L -supersolution) of $\text{PPDE}(G^\varepsilon)$, for some *fixed* $L > 0$. Assume

$$(G^\varepsilon, u^\varepsilon) \longrightarrow (G, u) \text{ as } \varepsilon \rightarrow 0, \text{ loc. unif.}$$

Then u is a viscosity L -subsolution (resp. supersolution) of PPDE with coefficient G .

Proofs are easy application of optimal stopping theory

Wellposedness under additional assumptions

Ekren, NT & Zhang 2012a obtained **existence and uniqueness** under

Assumption G2 Either one of the following conditions :

- (i) G convex in γ and uniformly elliptic,
- (ii) or, G is convex in (y, z, γ)
- (iii) or, $d \leq 2$

- proofs based on standard PDE theory applied to path-frozen equation...
- existence and comparison are combined

Our objective :

- avoid relying on PDE wellposedness results...
- take advantage of our richer class of test functions

Semilinear path-dependent PDEs

$$\begin{aligned}
 \text{(PPDE)} \quad & -\partial_t u - \frac{1}{2} \text{Tr}[\sigma \sigma^T(t, \omega) \partial_{\omega\omega}^2 u] - F(t, \omega, u, \sigma^T \partial_\omega u) = 0 \\
 & u_T = \xi
 \end{aligned}$$

In this case, we only need a subset of probability measures on Ω :

$$\mathcal{P}_L^0 := \{ \mathbb{P} \in \mathcal{P}_L : d\langle B \rangle_t = \sigma \sigma^T dt \}$$

Assumption σ continuous in t , Lipschitz in ω , unif. in t

Main result

Assumption $F(t, \omega, y, z)$ Lipschitz in (y, z) , uniformly in (t, ω) , and $F(t, \omega, 0, 0)$ bounded

Denote :

$$C_{2, \mathcal{P}_L^0}^0 := \left\{ u \in C^0(\Lambda) : \sup_{\mathbb{P} \in \mathcal{P}_L^0} \mathbb{E}^{\mathbb{P}} \left[\sup_{t \in [0, T]} |u_t|^2 \right] < \infty \right\}$$

Theorem (Ren, NT & Zhang 2014)

Let $u, v \in C_{2, \mathcal{P}_L^0}^0$ be viscosity subsolution and super solution, respectively, of PPDE. Then

$$u_T \leq v_T \implies u \leq v \text{ on } [0, T] \times \Omega$$

Purely probabilistic proof adapting ideas from Caffarelli and Cabre...



The Linear Case

Theorem (Ren, NT & Zhang 2014)

For $u \in C_{2, \mathbb{P}_0}^0$, the following are equivalent :

- u is a viscosity subsolution of $-\partial_t u - \frac{1}{2} \partial_{\omega\omega}^2 u \leq 0$
- u is a submartingale.

A similar statement holds for supersolutions.

- Comparison for the path-dependent heat equation follows immediately
- In the linear case, our notion of viscosity solutions is equivalent to the stochastic viscosity solutions of Bayraktar & Sirbu

\mathbb{L}^1 -type of regularity

Consider the \mathbb{L}^1 -distance on Ω :

$$d_1[(t, \omega), (t', \omega')] = |t - t'| + \int_0^T |\omega_{s \wedge t} - \omega'_{s \wedge t'}| ds + |\omega_t - \omega'_{t'}|$$

Assumption The nonlinearity G is d_1 -uniformly continuous in (t, ω) and uniformly Lipschitz in (y, s, γ) .

Theorem (Ren, NT & Zhang 2015)

Let u and v be bounded d_1 -uniformly continuous viscosity subsolution and supersolution of PPDE, respectively. If $u_T \leq v_T$, then $u \leq v$ on $[0, T] \times \Omega$

THANK YOU FOR YOUR ATTENTION