Numerical approximation of obliquely reflected BSDEs

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Outline

Reflected BSDEs - Markovian setting

- For $(b, \sigma) : \mathbb{R}^d \to \mathbb{R}^d \times M^d$ Lipschitz (σ may be degenerate) :

$$X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u$$

- 'Simply' reflected BSDEs on a boundary $l(X)$, (Y, Z, K)

$$Y_t = g(X_T) + \int_t^T f(X_t, Y_t, Z_t)\mathrm{d}t - \int_t^T (Z_t)'\mathrm{d}W_t + \int_t^T \mathrm{d}K_t$$

(C1) $Y_t \geq l(X_t)$ (constrained value process)

(C2) $\int_0^T \Big(Y_t - l(X_t)\Big)\mathrm{d}K_t = 0$ ("optimality" of K)

- Extension: doubly reflected BSDEs, reflected BSDEs in convex domain $\hookrightarrow$ *normal reflection*

Geometric framework

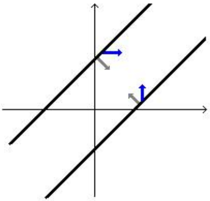
- Multidimensional process Y constrained in a domain $\mathcal{C}$ ($d \geq 2$)

$$\mathcal{C} = \{\, y \in \mathbb{R}^d | y^i \geq \max_j(y_j - c_{ij}) =: \mathcal{P}^i(y)\ ,\ 1 \leq i \leq d \,\}$$

with $c_{ii} = 0$, $\inf_{i \neq j} c_{ij} > 0$, $c_{ij} + c_{jk} > c_{ik}$

$\hookrightarrow \mathcal{P}$ (oblique projection) is L-lipschitz with $L > 1$ (euclidean norm)
for this talk: c_{ij} are constant coefficients

- example $d = 2$, oblique direction of reflection



Obliquely reflected BSDEs

- System of reflected BSDEs: for $1 \le i \le d$,

$$Y_t^i = g^i(X_T) + \int_t^T f^i(X_u, Y_u, Z_u^i)\mathrm{d}u - \int_t^T (Z_u^i)'\mathrm{d}W_u + \int_t^T \mathrm{d}K_s^i$$

(**C1**) $Y_t \in \mathcal{C}$ (constrained by K)

(**C2**) $\int_0^T \Big(Y_t^i - \mathcal{P}^i(Y_t)\Big)\mathrm{d}K_t^i = 0$ ('optimality' of K)

- Existence and uniqueness:

(H1): $f^i(y,z) = f^i(y^i, z^i)$, Hu & Tang 07, Hamadene & Zhang 08

(H2): $f^i(y,z) = f^i(y, z^i)$, C.-Elie-Kharroubi 11

Starting and Stopping problem (1)

Hamadene and Jeanblanc (01):

- Consider e.g. a power station producing electricity whose price is given by a diffusion process X: $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$
- Two modes for the power station:
  mode $\mathbf{1}$: operating, profit is then $f^1(X_t)\mathrm{d}t$
  mode $\mathbf{2}$: closed, profit is then $f^2(X_t)\mathrm{d}t$
  $\hookrightarrow$ switching from one mode to another has a cost: $c > 0$
- Management decide to produce electricity only when it is profitable enough.
- The management strategy is (θ_j, α_j) : θ_j is a sequence of stopping times representing switching times from mode α_{j-1} to α_j.

  $(a_t)_{0 \le t \le T}$ is the state process (the management strategy).

Starting and Stopping problem (2)

- Following a strategy a from t up to T, gives

$$J(a,t) = \int_t^T f^{a_s}(X_s)\mathrm{d}s - \sum_{j\geq 0} c\mathbf{1}_{\{t\leq\theta_j\leq T\}}$$

- The optimization problem is then (at $t = 0$, for $\alpha_0 = 1$)

$$Y_0^1 := \sup_a \mathbb{E}[J(a,0)]$$

At any date $t \in [0,T]$ in state $i \in \{1,2\}$, the value function is Y_t^i.

Solution

- Y is solution of a coupled optimal stopping problem

$$Y_t^1 = \operatorname{ess}\sup_{t \le \tau \le T} \mathbb{E}\left[\int_t^\tau f(1, X_s)\mathrm{d}s + (Y_\tau^2 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

$$Y_t^2 = \operatorname{ess}\sup_{t \le \tau \le T} \mathbb{E}\left[\int_t^\tau f(2, X_s)\mathrm{d}s + (Y_\tau^1 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

- The optimal strategy (θ_j^*, α_j^*) is given by

$$\theta_{j+1}^* := \inf\{s \ge \theta_j^* \mid Y_s^{\alpha_j^*} = \max_{i \in \{1,2\}} Y_s^i - c\}$$

$$\alpha_{j+1}^* := \mathbf{1} \text{ if } \alpha_j^* = 2\,, \ \textbf{or } \mathbf{2} \text{ if } \alpha_j^* = 1\,.$$

System of reflected BSDEs

Y is the solution of the following system of reflected BSDEs:

$$Y_t^i = \int_t^T f(i, X_s)\mathrm{d}s - \int_t^T (Z_s^i)'\mathrm{d}W_s + \int_t^T \mathrm{d}K_s^i \ , \ i \in \{1, 2\} \ ,$$

with (the coupling...)

$$Y_t^1 \geq Y_t^2 - c \text{ and } Y_t^2 \geq Y_t^1 - c \ , \ \forall t \in [0, T]$$

and ('optimality' of K)

$$\int_0^T \Big(Y_s^1 - (Y_s^2 - c) \Big) \mathrm{d}K_s^1 = 0 \text{ and } \int_0^T \Big(Y_s^2 - (Y_s^1 - c) \Big) \mathrm{d}K_s^2 = 0$$

From “Switching problem” to “switched BSDEs”

- “Switching” strategy $a = (\alpha_j, \theta_j)_j$ starting at (i, t)

  $N^a = \#\{k \in \mathbb{N}^* | \theta_k \leq T\}$ (number of switch)

- State process - cost process

  $a_s = \alpha_0 \mathbf{1}_{0 \leq s \leq \theta_0} + \sum_{j=1}^{N^a} \alpha_{j-1} \mathbf{1}_{\theta_{j-1} < s \leq \theta_j}$, $A^a_s := \sum_{j=1}^{N^a} c_{\alpha_{j-1}, \alpha_j} \mathbf{1}_{\theta_j \leq s \leq T}$

- “Switched” BSDE (following the strategy a) under **(H1)**

$$U^a_t = g^{a_T}(X_T) + \int_t^T f^{a_s}(X_s, U^a_s, V^a_s) \mathrm{d}s - \int_t^T V^a_s \mathrm{d}W_s - A^a_T + A^a_t$$

- Representation (a^*: optimal strategy) under **(H1)**

$$Y^i_t = \operatorname*{esssup}_a U^a_t = U^{a*}_t$$

Outline

Implicit Euler Scheme for BSDE

- We are given an equidistant grid $\pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\}$, define $h = T/n$.

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- The scheme: given the terminal condition $Y_n = g(X_T)$, the transition from step $i+1$ to i is

$$Y_i := \mathbb{E}_{t_i}[Y_{i+1} + hf(Y_i, Z_i)]$$

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$\hookrightarrow$ in practice: approximation of the forward process/ estimation of the conditional expectation.

Deriving the scheme (1/2) - Y part

On the equidistant grid $\pi = \{0 = t_0 < ... < t_i < ... < t_n = T\}$, with $h = T/n$.

- Start with:

$$Y_{t_i} + \int_{t_i}^{t_{i+1}} Z_s \mathrm{d}W_s = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(Y_s, Z_s)\mathrm{d}s \quad (1)$$

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- *For the Z-part:*
  Multiply (1) by $\Delta W_i := W_{t_{i+1}} - W_{t_i}$, take conditional expectation:

$$\mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s\Big] \simeq \mathbb{E}_{t_i}\big[\Delta W_i Y_{t_{i+1}}\big]$$

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$$\hookrightarrow \quad Z_i := \mathbb{E}_{t_i}[H_i Y_{i+1}] \quad \text{with} \quad H_i := h^{-1}\Delta W_i\ .$$

A scheme for the RBSDE: an example with $f = 0$

- RBSDE, Snell envelop of $l(X_t)$:

$$Y_t = l(X_T) - \int_t^T (Z_u)' \mathrm{d}W_u + \int_t^T \mathrm{d}K_s\,,\ Y_t \geq l(X_t)$$

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- Given a grid $\Re = \{0 = r_0 < ... < r_k < ... < r_\kappa = T\}$, $\bar{Y}$ discrete Snell envelop of $(l(X_{r_k}))_k$:

$$\tilde{Y}_{r_k} := \mathbb{E}\left[\bar{Y}_{r_{k+1}} \mid \mathcal{F}_{r_k}\right]$$
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- More general domain/reflection:

$$\bar{Y}_{r_k} := \tilde{Y}_{r_k} \vee l(X_{r_k}) \to \bar{Y}_{r_k} = \mathcal{P}(\tilde{Y}_{r_k})$$

Moonwalk scheme for the RBSDE

- Implicit Euler scheme for the “BSDE part” :

$$\tilde{Y}_i := \mathbb{E}\left[Y_{i+1} \mid \mathcal{F}_{t_i}\right] + hf(\tilde{Y}_i, Z_i)$$
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- Taking into account the reflection

$$Y_i := \tilde{Y}_i \mathbf{1}_{t_i \notin \Re} + \mathcal{P}(\tilde{Y}_i)\mathbf{1}_{t_i \in \Re}$$

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- and terminal condition $Y_n = \tilde{Y}_n := g(X_{t_n})$.

Previous results and known difficulties

- In C.-Elie-Kharroubi 12, the convergence of the above method with a polynomial rate is proved under the following assumption on f

$$f^{\ell}(y,z) = f^{\ell}(y^{\ell}) \; .$$

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- The goal of this work is to prove convergence with polynomial rate under **(H2)** i.e. when f satisfies

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- In CEK12, the absence of z in f comes really from the final approximation i.e. the scheme: stability issue.

Outline

Stability - perturbation approach

Recall the scheme,

$$\tilde{Y}_i := \mathbb{E}\left[Y_{i+1} + hf(\tilde{Y}_i, Z_i) \mid \mathcal{F}_{t_i}\right]$$
$$Z_i := \mathbb{E}\left[Y_{i+1} H_i' \mid \mathcal{F}_{t_i}\right]$$
$$Y_i := \tilde{Y}_i \mathbf{1}_{t_i \notin \Re} + \mathcal{P}(\tilde{Y}_i) \mathbf{1}_{t_i \in \Re}$$

We consider a pertubed (for the Y-part) scheme:

$$\tilde{\mathcal{Y}}_i := \mathbb{E}\left[\mathcal{Y}_{i+1} + hf(\tilde{\mathcal{Y}}_i, \mathcal{Z}_i) + \zeta_i \mid \mathcal{F}_{t_i}\right]$$
$$\mathcal{Z}_i := \mathbb{E}\left[\mathcal{Y}_{i+1} H_i' \mid \mathcal{F}_{t_i}\right]$$
$$\mathcal{Y}_i := \tilde{\mathcal{Y}}_i \mathbf{1}_{t_i \notin \Re} + \mathcal{P}(\tilde{\mathcal{Y}}_i) \mathbf{1}_{t_i \in \Re}$$

set $\delta Y := \mathcal{Y} - Y$, $\delta\tilde{Y} := \tilde{\mathcal{Y}} - \tilde{Y}$ and $\delta Z := \mathcal{Z} - Z$.

Stability - Definition

The scheme is said to be L^2-stable if

$$\max_i \mathbb{E}\Big[|\delta Y_i|^2 + |\delta \tilde{Y}_i|^2\Big] + \frac{1}{\alpha(\kappa)} \sum_i h\mathbb{E}\big[|\delta Z_i|^2\big] \leq C \sum_{i=0}^{n-1} h\mathbb{E}\Big[\frac{1}{h^2}|\zeta_i|^2\Big]$$

i.e. the overall error is the sum of the error done at each step. α positive non decreasing function.

- (Sufficient Condition for L^2-Stability) If f is Lipschitz-continuous and $f^\ell(y,z) = f^\ell(y,z^\ell)$, then the scheme is L^2-stable with $\alpha(\kappa) = \kappa$.

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$$\leq (1 + Ch)\mathbb{E}_{t_{i-1}}\big[|\delta Y_i|_2^2\big] \; + \; C\frac{|\zeta_i|^2}{h}$$

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$$\leq (1 + Ch)\mathbb{E}_{t_{i-1}}\big[|\delta Y_i|_2^2\big] \; + \; C\frac{|\zeta_i|^2}{h}$$

$\hookrightarrow$ But $|\delta Y_i|_2 := |\mathcal{P}(\tilde{Y}_i) - \mathcal{P}(\tilde{\mathcal{Y}}_i)|_2 \leq L|\tilde{Y}_i - \tilde{\mathcal{Y}}_i|_2 =: L|\delta\tilde{Y}_i|_2$, $L > 1$

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- Nevertheless, we observe that $\mathcal{P}$ is 1-Lipschitz for $|\cdot|_\infty$ and use:

$$|\delta \tilde{Y}_{i-1}|_\infty^2 \leq |\delta Y_i|_\infty^2 + \sum_{j=1}^{d} \big(2\delta \tilde{Y}_{i-1}^j(\delta \tilde{Y}_{i-1}^j - \delta Y_i^j) - |\delta Y_i^j - \delta \tilde{Y}_{i-1}^j|^2\big)\mathbf{1}_{\{j=j^*\}}$$

where $j^* = \min\{j \mid |\delta \tilde{Y}_{i-1}^j| = \max_k |\delta \tilde{Y}_{i-1}^k|\}$.

Key object - Discretely reflected BSDEs (dRBSDEs)

Given a grid $\Re = \{0 = r_0 < ... < r_k < ... < r_\kappa = T\}$,
a triplet $(Y^d, \tilde{Y}^d, Z^d)$ satisfying

$$Y^d_T = \tilde{Y}^d_T := g(X_T)$$

and, for $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\begin{cases} \tilde{Y}^d_t &= Y^d_{r_{j+1}} + \int_t^{r_{j+1}} f(X, \tilde{Y}^d, Z^d)\mathrm{d}u - \int_t^{r_{j+1}} (Z^d_u)'\mathrm{d}W_u \quad , \\ Y^d_t &= \tilde{Y}^d_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(\tilde{Y}^d_t)\mathbf{1}_{\{t \in \Re\}}. \end{cases}$$

Example: simply reflected on $l(X)$, set $f = 0$, $g = l$... dRBSDE is the discrete Snell envelop of $l(X_r)_{r \in \Re}$

Strategy: first step

I. Approximate the discretely reflected BSDE:

1. *truncation error*: regularity of the discretely RBSDE (minimal Lipschitz condition, σ can be degenerate)

   $\hookrightarrow$ Use a representation result for $Z^{\Re}$ based on Malliavin Calculus.

2. *global error*: Control via stability result.

   $\hookrightarrow$ See discretely RBSDE as a pertubed scheme.

We obtain

$$\mathcal{E}rr(Y^{\Re}, Y^{\pi}) + \mathcal{E}rr(Z^{\Re}, Z^{\pi}) \leq C(\kappa^{\frac{1}{2}} h^{\frac{1}{2}} + h^{\frac{1}{2}}).$$

Strategy: second step

II. Extension to the continuously reflected BSDE:

1. Discretely RBSDE is a good proxy for continuously BSDEs:

$$\mathcal{E}rr(Y, Y^{\Re}) + \mathcal{E}rr(Z, Z^{\Re}) \leq C\kappa^{-\frac{1}{2}}$$

2. Use the scheme for the discretely RBSDE setting $\kappa = O(h^{-\frac{1}{2}})$ to obtain :

$$\mathcal{E}rr(Y, Y^{\pi}) + \mathcal{E}rr(Z, Z^{\pi}) \leq Ch^{\frac{1}{4}} .$$

Error between continuous and discrete reflection

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$$0 \leq (Y_t)^i - (Y_t^{\Re})^i \leq U_t^a - U_t^{\bar{a}}\,, \quad \bar{a} \text{ is the projection of strategy } a \text{ on } \Re$$

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2. Under **(H2)**: no comparison...
$\hookrightarrow$ force comparison! Introduce $(\check{Y}, \check{Z})$ "max" of (Y, Z) and $(Y^{\Re}, Z^{\Re})$ and then

$$|(Y_t)^i - (Y_t^{\Re})^i| \leq |(\check{Y}_t)^i - (Y_t)^i| + |(\check{Y}_t)^i - (Y_t^{\Re})^i|$$

Conclude by using 1.

Concluding Remarks

- New convergence result when f depends on z
- Based on a careful study of the scheme L^2 stability
- Empirical Scheme:
  • "Low" regularity $\hookrightarrow$ Use 1) quantization or 2) regression method.
  • Full convergence for 1) follows "easily" from key stability result for the scheme.