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Infinite-dimensional stochastic differential equations and tail σ-fields

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Outline

1. Introduction

Infinite dimensional stochastic differential equations (ISDEs)

- $\{B^i\}_{i\in\mathbb{N}}$ are independent d-dimensional Brownian motions.
- $\Phi = \Phi(x)$; free potential.
- $\Psi = \Psi(x, y)$; interaction potential.

We study ISDEs of $\mathbf{X} = (X^i)_{i\in\mathbb{N}} \in C([0,\infty); (\mathbb{R}^d)^{\mathbb{N}})$:

ISDE (1)

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla_x\Phi(X_t^i)dt - \frac{\beta}{2}\sum_{j\neq i}^{\infty}\nabla_x\Psi(X_t^i, X_t^j)dt$$

$$(X_0^i)_{i\in\mathbb{N}} = \mathbf{s} = (s_i)_{i\in\mathbb{N}}$$

- Existence (solutions, strong solutions)
- Uniqueness (in distribution, pathwise)

1. Introduction

Related results

1. Lang 1977,1978: Ψ : smooth, with compact support
2. Fritz 1987: singular interaction
3. T. 1996, Fradon-Roelly-T. 2002: Ψ : with hard core
4. Osada 2012: Ψ : Ruelle's class, logarithmic (Dyson, Ginibre)
   Existence of solutions
5. Honda-Osada 2015 : logarithmic (Bessel)
   Existence and uniqueness of solutions
6. Osada-T. (arXiv1408.0632): logarithmic (Airy)
   Existence and uniqueness of solutions

2. Preliminaries

General ISDEs

Let $T \in \mathbb{N} \equiv \{1, 2, \dots\}$ and $S = \mathbb{R}^d$.

$\mathbf{W} = C([0,T]; S^{\mathbb{N}})$,

$\mathbf{W}^{sol}$ is a Borel subset of $\mathbf{W}$,

$$\sigma^i : \mathbf{W}^{sol} \to C([0,T]; \mathbb{R}^{d^2}), \quad b^i : \mathbf{W}^{sol} \to C([0,T]; \mathbb{R}^d).$$

ISDE(2)

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt$$

$$\mathbf{X} \in \mathbf{W}_{\mathbf{s}}^{sol} = \{\mathbf{X} \in \mathbf{W}^{sol} : \mathbf{X}_0 = \mathbf{s}\}$$

Assumption (B1)

ISDE (2) has a solution $\mathbf{X}$.

2. Preliminaries

Definition of solutions

Let $\mathbf{X}$ be a solution of ISDE(2) with the Brownian motion $\mathbf{B}$.

Definition

1. We call $(\mathbf{X}, \mathbf{B})$ a strong solution of ISDE(2) if $\mathbf{X}$ is a function of $\mathbf{B}$ defined for a.s. $\mathbf{B}$. In this case we set $\mathbf{X} = \mathbf{X}(\mathbf{B})$.
2. We call ISDE(2) has a unique strong solution if a strong solution of ISDE(2) exists and $\mathbf{X} = \mathbf{X}'$ a.s. for any pair of strong solutions $(\mathbf{X}, \mathbf{B})$ and $(\mathbf{X}', \mathbf{B})$.
3. We call the pathwise uniqueness of solutions for ISDE (2) holds if $\mathbf{X} = \mathbf{X}'$ a.s. for any solutions $\mathbf{X}$ and $\mathbf{X}'$ on the same probability space with the same Brownian motion $\mathbf{B}$.
4. We call the uniqueness in law of solutions for ISDE (2) holds if the distributions of any solutions $\mathbf{X}$ and $\mathbf{X}'$ coincide.
5. We call the strong uniqueness holds for ISDE(2) if 1 – 4 are satisfied.

2. Preliminaries

The infinite system of finite-dimensional SDEs

For $\mathbf{X} \in \mathbf{W}$ and $m \in \mathbb{N}$ we put

$$\mathbf{X}^m = (X^1, X^2, \dots, X^m), \quad \mathbf{X}^{m*} = (X^{m+1}, X^{m+2}, \dots).$$

For given $\mathbf{X} \in \mathbf{W}_{\mathbf{s}}^{sol}$, we consider the SDE on S^m of

$$\mathbf{Y}^m = (Y^{m,1}, Y^{m,2}, \dots, Y^{m,m}).$$

SED(3)

$$dY_t^{m,i} = \sigma^i((\mathbf{Y}^m.\mathbf{X}^{m*}))_t dB_t^i + b^i((\mathbf{Y}^m.\mathbf{X}^{m*}))_t dt$$

$$(\mathbf{Y}^m, \mathbf{X}^{m*}) \in \mathbf{W}_{\mathbf{s}}^{sol}$$

We call the sequence $(\mathbf{Y}^m)_{m\in\mathbb{N}}$ an infinite system of inite dimensional SDEs associated with ISDE(2).

2. Preliminaries

IFC (infinite system of finite-dimensional SDEs with consistency)

Assumption (B2)

For each $\mathbf{X} \in \mathbf{W}_{\mathbf{s}}^{sol}$, SDE(3) has a strong solution $\mathbf{Y}^m$, and the pathwise uniqueness for solutions holds for each $m \in \mathbb{N}$.

Put

$$F_{\mathbf{s}}^m(\mathbf{X}, \mathbf{B}) = (\mathbf{Y}^m, \mathbf{X}^{m*}) = (Y^{m,1}, \dots, Y^{m,m}, X^{m+1}, X^{m+2}, \dots).$$

Definition (IFC solution)

A probability measure $\overline{P}_{\mathbf{s}}$ on $\mathbf{W} \times \mathbf{W_0}$ is called an IFC solution for ISDE(2) if $\overline{P}_{\mathbf{s}}$ satisfies

- $\overline{P}_{\mathbf{s}}(\mathbf{W}_{\mathbf{s}}^{sol} \times \mathbf{W_0}) = 1$
- $\overline{P}_{\mathbf{s}}(\mathbf{B} \in \cdot) = P_{Br}^{\infty}(\cdot)$
- $F_{\mathbf{s}}^{\infty}(\mathbf{X}, \mathbf{B}) = \lim_{m \to \infty} F_{\mathbf{s}}^m(\mathbf{X}, \mathbf{B})$ in $\mathbf{W}^{sol}$ under $\overline{P}_{\mathbf{s}}$

2. Preliminaries

where

$$F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B}) = \lim_{m\to\infty} F_{\mathbf{s}}^{m}(\mathbf{X},\mathbf{B}) \quad \text{in } \mathbf{W}^{sol} \text{ under } \overline{P}_{\mathbf{s}}$$

means that for $\forall i \in \mathbb{N}$ for $\overline{P}_{\mathbf{s}}$-a.s. $(\mathbf{X},\mathbf{B})$

- $\lim_{m\to\infty} F_{\mathbf{s}}^{m,i}(\mathbf{X},\mathbf{B}) = F_{\mathbf{s}}^{\infty,i}(\mathbf{X},\mathbf{B})$
- $\lim_{m\to\infty} \int_0^{\cdot} \sigma^i(F_{\mathbf{s}}^{m}(\mathbf{X},\mathbf{B}))_u dB_u^i = \int_0^{\cdot} \sigma^i(F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B}))_u dB_u^i$
- $\lim_{m\to\infty} \int_0^{\cdot} b^i(F_{\mathbf{s}}^{m}(\mathbf{X},\mathbf{B}))_u du = \int_0^{\cdot} b^i(F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B}))_u du$

in $C([0,T],\mathbb{R}^d)$.

2. Preliminaries

<u>Solutions and IFC solutions</u>

Assume (B1) and (B2).

Let $\overline{P}_{\mathbf{s}}$ be the distribution of a solution of ISDE (2).

Facts

1. $F_{\mathbf{s}}^{m,i}(\mathbf{X},\mathbf{B}) = F_{\mathbf{s}}^{m+1,i}(\mathbf{X},\mathbf{B}) \quad \forall m \in \mathbb{N}, \forall i = 1,2,\ldots,m.$
2. $\overline{P}_{\mathbf{s}}$ is an IFC solution for ISDE (2).
3. $(F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B}),\mathbf{B}) = (\mathbf{X},\mathbf{B}) \quad \overline{P}_{\mathbf{s}}$-a.s. $(\mathbf{X},\mathbf{B})$.

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Assume (B2).

Let $\overline{P}_{\mathbf{s}}$ be an IFC solution for ISDE (2).

Fact

4. $F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B})$ is a solution of ISDE (2) under $\overline{P}_{\mathbf{s}}$ with $\mathbf{B}$.

Preliminaries

Tail σ-fields

$$F_{\mathbf{s}}^{\infty} : \mathbf{W_s} \times \mathbf{W_0} \to \mathbf{W_s}$$

Fact

5. The map $F_{\mathbf{s}}^{\infty}$ is $\mathcal{T}_{path}(S^{\mathbb{N}}) \times \mathcal{B}(\mathbf{W_0})$ measurable.

Here $\mathcal{T}_{path}(S^{\mathbb{N}})$ is the tail σ-field of $\mathbf{W}$ defined by

$$\mathcal{T}_{path}(S^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma(\mathbf{X}^{m*}), \text{ where } \mathbf{X}^{m*} = (X^i)_{i=m+1}^{\infty}.$$

For a probability measure P such that $\mathcal{T}_{path}(S^{\mathbb{N}})$ is trivial, that is,

$$P(A) \in \{0, 1\}, \forall A \in \mathcal{T}_{path}(S^{\mathbb{N}}),$$

we set

$$\mathcal{T}_{path}^{[1]}(S^{\mathbb{N}}; P) = \{A \in \mathcal{T}_{path}(S^{\mathbb{N}}); P(A) = 1\}.$$

3. Theorem 1 (First tail theorem)

$\overline{P}_{\mathbf{s},\mathbf{B}}(\cdot) = \overline{P}_{\mathbf{s}}(\cdot|\mathbf{B})$: the regular conditional probability.

Assumptions (B3) – (B5)

(B3) $\mathcal{T}_{path}(S^{\mathbb{N}})$ is $\overline{P}_{\mathbf{s},\mathbf{B}}$-trivial for P^{∞}_{Br}- a.s. $\mathbf{B}$.

(B4) $\mathcal{T}^{[1]}_{path}(S^{\mathbb{N}};\overline{P}_{\mathbf{s},\mathbf{B}}) = \mathcal{T}^{[1]}_{path}(S^{\mathbb{N}};\overline{P'}_{\mathbf{s},\mathbf{B}})$ for P^{∞}_{Br}- a.s. $\mathbf{B}$.

(B5) $\mathcal{T}^{[1]}_{path}(S^{\mathbb{N}};\overline{P}_{\mathbf{s},\mathbf{B}})$ is independent of $\overline{P}_{\mathbf{s},\mathbf{B}}$ for P^{∞}_{Br}- a.s. $\mathbf{B}$.

Theorem 1 (First tail theorem)

1. (B.1)–(B.3) $\Rightarrow$ ISDE (2) has a strong solution.
2. (B.1)–(B.4) $\Rightarrow$ Strong solutions $\mathbf{X}$ and $\mathbf{X}'$ satisfy $\mathbf{X} = \mathbf{X}'$ a.s.
3. (B.1)–(B.5) $\Rightarrow$ Strong uniqueness holds for ISDE(2).

4. Theorem 2

Unlabeled configuration space on S

$$\mathfrak{S} = \{\mathfrak{s} = \sum_j \delta_{s_j} : \mathfrak{s}(K) < \infty \quad \forall K : compact\}$$

$\mathfrak{S}$ is Polish with the vague topology. Set

$$\mathfrak{S}_{\text{s.i.}} = \{\mathfrak{s} \in \mathfrak{S} : \mathfrak{s}(S) = \infty,\ \mathfrak{s}(\{s\}) \in \{0,1\},\ \forall s \in S\}.$$

Tale σ-field $\mathcal{T}(\mathfrak{S})$ on $\mathfrak{S}$ is given by

$$\mathcal{T}(\mathfrak{S}) = \bigcap_{r=1}^{\infty} \sigma(\pi_r^c)$$

where $\pi_r^c(\mathfrak{s})(\cdot) = \mathfrak{s}(\cdot \cap S_r^c), \quad S_r = \{s \in S : |s| \le r\}.$

4. Theorem 2

A labeled map on $\mathfrak{s}$ and $C([0,T],\mathfrak{S}_{\mathrm{s.i.}})$

A map $\mathfrak{l}:\mathfrak{S}\to S^{\mathbb{N}}$ given as

$$\mathfrak{l}(\mathfrak{s})=\mathbf{s}=(s_j)_{j\in\mathbb{N}},\ \text{for } \mathfrak{s}=\textstyle\sum_{j=1}^{\infty}\delta_{s_j}\in\mathfrak{S}$$

is called a label.

For a label $\mathfrak{l}$, we can deternine the map $\mathfrak{l}_{path}$

$$\mathfrak{l}_{path}:C([0,T],\mathfrak{S}_{\mathrm{s.i.}})\to C([0,T],S^{\mathbb{N}})$$

such that

$$\mathfrak{l}(\mathfrak{X})_0=\mathbf{X_0}=(X_0^j)_{j\in\mathbb{N}},\ \text{for } \mathfrak{X}=\textstyle\sum_{j=1}^{\infty}\delta_{X^m}\in C([0,T],\mathfrak{S}_{\mathrm{s.i.}})$$

For a prob. meas. P_{μ} on $C([0,T],\mathfrak{S})$ with $\mathrm{P}_{\mu}\circ\mathfrak{X}_0^{-1}=\mu$.

$$\mu^{\mathfrak{l}}=\mu\circ\mathfrak{l}^{-1},\quad \mathbf{P}_{\mu^{\mathfrak{l}}}=\mathrm{P}_{\mu}\circ\mathfrak{l}_{path}^{-1}$$

4. Theorem 2

Assumptions (C1) – (C4)

(C1) $\mathcal{T}(\mathfrak{S})$ is μ-trivial.

(C2) $\mathsf{P}_\mu \circ \mathfrak{X}_t{}^{-1} \prec \mu$, $\forall t \in [0, T]$

(C3) $\mathsf{P}_\mu(C([0, T], \mathfrak{S}_{\mathrm{s.i.}})) = 1$

(C.4) $\mathsf{P}_\mu(\bigcap_{r=1}^{\infty}\{m_r(\mathfrak{X}) < \infty\}) = 1$
where $m_r(\mathfrak{X}) = \inf\{m \in \mathbb{N} : X_t^n \in S_r^c, \ \forall t \in [0, T], \ \forall n > m\}$.

Theorem 2 (Second tail theorem)

Assume (B2). Suppose that there exists P_μ satisfying (C1) – (C4), and

$$\mathbf{P}_{\mu^{\mathfrak{l}}}(F_{\mathfrak{s}}^{\infty}(\mathbf{X}, \mathbf{B}) = \mathbf{X}) = 1.$$

Then (B1) – (B5) holds for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{s}$.

4. Theorem 2

$$\begin{array}{ccccccc} \mathcal{T}(\mathsf{S}) & \xrightarrow{\text{Step I}} & \tilde{\mathcal{T}}_{path}(\mathsf{S}) & \xrightarrow{\text{Step II}} & \tilde{\mathcal{T}}_{path}(S^{\mathbb{N}}) & \xrightarrow{\text{Step III}} & \mathcal{T}_{path}(S^{\mathbb{N}}) \\ \mu & & \mathsf{P}_{\mu} & & \mathbf{P}_{\mu^{\mathsf{l}}} & & \overline{P}_{\mathbf{s},\mathbf{B}} \end{array}$$

Here, $\tilde{\mathcal{T}}_{path}(\mathfrak{S})$ is the cylindrical tail σ-field on $C([0,T],\mathfrak{S})$ defined as

$$\tilde{\mathcal{T}}_{path}(\mathfrak{S})) = \bigvee_{\mathbf{t}=(t_1,t_2,\dots,t_n),n\in\mathbb{N}} \bigcap_{r=1}^{\infty} \sigma[\pi_r^c(\mathfrak{X}_{t_i}), 1 \le i \le n].$$

and $\tilde{\mathcal{T}}_{path}(S^{\mathbb{N}})$ is the cylindrical tail σ-field on $C([0,T],S^{\mathbb{N}})$ defined as

$$\tilde{\mathcal{T}}_{path}(S^{\mathbb{N}}) = \bigvee_{\mathbf{t}=(t_1,t_2,\dots,t_n),n\in\mathbb{N}} \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}_{t_i}^{m*}, 1 \le i \le n].$$

5. Applications

ISDE (1)

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla_x \Phi(X_t^i)dt - \frac{\beta}{2}\sum_{j\neq i}^{\infty} \nabla_x \Psi(X_t^i, X_t^j)dt$$

$$(X_0^i)_{i\in\mathbb{N}} = (s_i)_{i\in\mathbb{N}} = \mathsf{s}$$

- Construction of $\mathfrak{S}$-valued process $\mathfrak{X}_t = \{X_t^i\}_{i\in\mathbb{N}}$ by Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ associated with a (Φ, Ψ)-quasi-Gibbs state μ. [Osada AOP 2013].
- Existence of solutions of ISDE(1) related to the logarithmic derivative of the Campbell measure $\mu^{[1]}$ of μ [Osada PTRF 2012].

5. Applications

Uniqueness of strong solutions

Let μ be a quasi-Gibbs state.

Condition (A1)

ISDE (1) has a solution $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}}$ for μ^{l}-a.s. $\mathbf{s}$.

Condition (A2)

The distribution $\overline{P}_{\mathbf{s}}$ of $(\mathbf{X}, \mathbf{B})$ is an IFC solution of ISDE (1) for μ^{l}-a.s. $\mathbf{s}$.

We can check Condition (A2) for interaction of Ruelle's class, and logarithmic interaction (Dyson, Bessel, Airy, Ginibre).

Applications

Condition (A3)

$\mathsf{P}_{\mu} \circ \mathfrak{X}_t^{-1}$ for $\forall t > 0$. (μ-absolute continuity condition)

Condition (A4)

The tail σ-field $\mathcal{T}(\mathfrak{S})$ is μ-trivial

Theorem 3

Assume (A1) – (A4). Then, for μ^{l}-a.s. **s**, ISDE (1) has a strong solution satisfying the μ-absolute continuity condition, and strong uniqueness holds for ISDE(1) with the μ-absolutely continuity condition.

5. Applications

Let $\mu^{\mathfrak{a}}_{Tail}$ be a regular conditional probability measure by the tail σ-field defined by

$$\mu^{\mathfrak{a}}_{Tail} = \mu(\cdot|\mathcal{T}(\mathfrak{S})(\mathfrak{a}).$$

Then μ can be decomposed as

$$\mu(\cdot) = \int_{\mathfrak{S}} \mu^{\mathfrak{a}}_{Tail}(\cdot)\mu(d\mathfrak{a}).$$

We denote the conditions (A1)–(A3) for $\mu^{\mathfrak{a}}_{Tail}$ by (A1-$\mathfrak{a}$) – (A3-$\mathfrak{a}$).

Theorem 4

Assume (A1-$\mathfrak{a}$) – (A3-$\mathfrak{a}$). Then, for μ-a.s. $\mathfrak{a}$, for $(\mu^{\mathfrak{a}}_{Tail})^{\mathsf{l}}$-a.s. $\mathbf{s}$, ISDE (1) has a strong solution satisfying the $\mu^{\mathfrak{a}}_{Tail}$-absolute continuity condition, and strong uniqueness holds for ISDE(1) with the $\mu^{\mathfrak{a}}_{Tail}$-absolutely continuity condition.

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