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The circular unitary ensemble and the Riemann zeta function: the microscopic landscape

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Joint work with Reda Chhaibi and Ashkan Nikeghbali

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Introduction, presentation of the main setting

There are many different models of random matrices which can be studied. Here are two classical examples:

- The *Gaussian Unitary Ensemble*, for which the entries are complex Gaussian variables.
- The *Circular Unitary Ensemble*, corresponding to a random unitary matrix following the Haar (i.e. uniform) measure on a unitary group.

In this presentation, we study convergence results on the characteristic polynomial of a random matrix following the Circular Unitary Ensemble (CUE). We then state some conjectures on the Riemann zeta function, related to these results.

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In this presentation, we study convergence results on the characteristic polynomial of a random matrix following the Circular Unitary Ensemble (CUE). We then state some conjectures on the Riemann zeta function, related to these results.

More precisely, our seting is given as follows:

- We consider, for $n \geq 1$, a Haar-distributed matrix u_n on the unitary group $U(n)$.
- We denote by $(\lambda_k^{(n)})_{1 \leq k \leq n}$ the eigenvalues of u_n, ordered counterclockwise starting from 1: recall that all the eigenvalues have modulus 1.
- For $1 \leq k \leq n$, we denote by $\theta_k^{(n)} \in [0, 2\pi)$ the argument of $\lambda_k^{(n)}$. We extend by periodicity the notation to all $k \in \mathbb{Z}$.

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- The characteristic polynomial Z_n of u_n is given by

$$Z_n(z) := \det(zI_n - u_n) = \prod_{k=1}^{n} (z - \lambda_k^{(n)}).$$

- One can prove that for $|z| < 1$, $n \to \infty$, $Z_n(z)$ converges in law to a limiting random variable (consequence of a result by Diaconis and Shahshahani on the distribution of eigenvalues).
- Such a convergence does not hold for $|z| \geq 1$. For $|z| = 1$, Keating and Snaith have proven that $\log |Z_n(z)| / \sqrt{\log n}$ converges to a Gaussian random variable.

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Statement of the main result

In our paper, we consider the ratio:

$$\xi_n(z) := \frac{Z_n(e^{2i\pi z/n})}{Z_n(1)}.$$

We show the following result:

Theorem

There exists a random holomorphic function $(\xi_\infty(z))_{z\in\mathbb{C}}$ such that $(\xi_n(z))_{z\in\mathbb{C}}$ converges in law to $(\xi_\infty(z))_{z\in\mathbb{C}}$ when $n \to \infty$, in the space of continuous functions from $\mathbb{C}$ to $\mathbb{C}$, endowed with the topology of uniform convergence on compact sets.

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Some properties of ξ_∞:

- All the zeros of ξ_∞ are real.
- The set E of the zeros of ξ_∞ is a *determinantal sine-kernel point process*, i.e. for $m \geq 1$, f nonnegative and measurable from $\mathbb{R}^m$ to $\mathbb{R}$,

$$\mathbb{E}\left[\sum_{x_1 \neq \cdots \neq x_m \in E} f(x_1, \ldots, x_n)\right]$$

$$= \int_{\mathbb{R}^m} f(y_1, \ldots, y_m) \rho_m(y_1, \ldots, y_m) dy_1 \ldots dy_m$$

where

$$\rho_m(y_1, \ldots, y_m) = \det\left(\frac{\sin(\pi(y_p - y_q))}{\pi(y_p - y_q)}\right)_{1 \leq p,q \leq m}.$$

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$$\rho_m(y_1, \dots, y_m) = \det\left(\frac{\sin(\pi(y_p - y_q))}{\pi(y_p - y_q)}\right)_{1 \leq p,q \leq m}.$$

- The function ρ_m is called the *m-point correlation function* of the point process E.
- ρ_1 is identically equal to 1, so E has the same 1-point correlation as a Poisson point process of intensity 1.
- The 2-point correlation function is smaller than or equal to 1, so the points of E tend to repel each other.
- This repulsion corresponds, at the limit when the dimension goes to infinity, to the general phenomenon of repulsion for the eigenvalues of unitary ensembles.

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The notion of virtual isometry

A classical result (due to Dyson up to technicalities) about the CUE is the following: if we multiply the eigenangles of u_n by $n/2\pi$, then the corresponding point process

$$E_n = \{y_k^{(n)} := n\theta_k^{(n)}/2\pi, k \in \mathbb{Z}\}$$

weakly converges to a determinantal sine-kernel process E.

- The convergence of E_n toward E is a weak convergence: if we want to get a strong convergence, we need to define all the matrices $(u_n)_{n\geq 1}$ on the same probability space.

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- In articles with Bourgade, Maples and Nikeghbali, we study a particular coupling of the dimensions n, in such a way that an almost sure convergence occurs.
- In our construction, the sequence $(u_n)_{n\geq 1}$ is almost surely a so-called *virtual isometry*.
- A virtual isometry is a sequence $(u_n)_{n\geq 1}$ such that for all $n \geq 1$,

$$\text{rank}(u_{n+1} - \text{Diag}(u_n, 1)) = \min_{u \in U(n)} \text{rank}(u_{n+1} - \text{Diag}(u, 1)).$$

- If $(u_n)_{n\geq 1}$ is a virtual isometry, one can write $u_{n+1} = r_{n+1}\,\text{Diag}(u_n, 1)$, where r_{n+1} is a complex reflection, in the sense that it is unitary and the rank of $r_{n+1} - I_{n+1}$ is at most 1.

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- There exists a unique measure on the space of virtual isometries under which for all $n \geq 1$, u_n is Haar-distributed.
- Borodin, Olshanski and Vershik have proven the almost sure convergence of E_n under this measure: almost surely, for all $k \in \mathbb{Z}$, $y_k^{(n)}$ tends to a limit y_k when n goes to infinity.
- The process $(y_k)_{k \in \mathbb{Z}}$ is a determinantal sine-kernel process.
- In papers with Bourgade, Maples, Nikeghbali, we have a more direct proof and we give a rate of convergence: for all $\alpha > 1/3$, one has almost surely, for $k \in \mathbb{Z}$, $n \geq k^4$,

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Application to the characteristic polynomial

- An elementary computation gives the following formula:

$$\xi_n(z) = e^{i\pi z} \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}} \right).$$

- The infinite product just above is not absolutely convergent. The absolute convergence occurs when we regroup the term of index k with the term of index $1-k$, for $k \geq 0$.

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- Since $y_k^{(n)} \to y_k$ for $n \to \infty$, it is natural to expect the following result

$$\xi_n(z) \underset{n\to\infty}{\longrightarrow} e^{i\pi z} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{y_k}\right) =: \xi_\infty(z).$$

- By using some estimates on the distribution of the points of a determinantal sine-kernel process, deduced from results by Costin, Lebowitz, Meckes and Soshnikov, one shows that the previous infinite product converges if we regroup the terms of indices k and $1-k$.

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- By using the estimate of $y_k^{(n)} - y_k$ given by our results with Maples and Nikeghbali, we are able to prove that almost surely, ξ_n converges to ξ_∞, uniformly on compact sets of $\mathbb{C}$.
- If we forget about the coupling between the different dimensions, we get the weak convergence stated at the beginning of the talk.
- The zeros of this function are exactly $(y_k)_{k\in\mathbb{Z}}$.

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Some properties of the function ξ_∞

- For any $z \in \mathbb{C}$, the random variable $\xi_\infty(z)$ has no moment of order 1 or higher.
- However, if we take ratios of ξ_∞ at different points outside the real line, with the same number of factors at the numerator and the denominator, the moments can be computed.
- More precisely, for all $z_1, \dots, z_k, z'_1, \dots, z'_k \in \mathbb{C} \backslash \mathbb{R}$ such that $z_i \neq z'_j$ for $1 \leq i, j \leq n$, we have

$$\det\left(\frac{1}{z_i - z'_j}\right)_{i,j=1}^k \mathbb{E}\left(\prod_{j=1}^k \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right) = \det\left(\frac{1}{z_i - z'_j}\mathbb{E}\left(\frac{\xi_\infty(z'_j)}{\xi_\infty(z_i)}\right)\right)_{i,j=1}^k$$

where

$$\mathbb{E}\left(\frac{\xi_\infty(z')}{\xi_\infty(z)}\right) = \begin{cases} 1 & \text{if } \Im(z) > 0 \\ e^{i2\pi(z'-z)} & \text{if } \Im(z) < 0 \end{cases}$$

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- The logarithmic derivative of ξ_∞ has moments of any order, at any point on $\mathbb{C} \backslash \mathbb{R}$.
- We have an explicit (quite complicated) combinatorial formula for the joint moments of ξ'_∞ / ξ_∞.
- In particular, we get, for $z \notin \mathbb{R}$:

$$\mathbb{E}\left[\frac{\xi'_\infty(z)}{\xi_\infty(z)}\right] = 2i\pi \mathbb{1}_{\Im(z)<0}$$

and

$$\mathbb{E}\left[\left|\frac{\xi'_\infty(z)}{\xi_\infty(z)}\right|^2\right] = 4\pi^2 \mathbb{1}_{\Im(z)<0} + \frac{1 - e^{-4\pi|\Im(z)|}}{4(\Im(z))^2}.$$

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- We have also proven that the order of ξ_∞ as an holomorphic function is one.
- More precisely, there exists a.s. a random $C > 0$, such that for all $z \in \mathbb{C}$,

$$|\xi_\infty(z)| \leq e^{C|z|\log(2+|z|)}.$$

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- On the other hand, there exists a.s. a random $c > 0$ such that for all $x \in \mathbb{R}$,

$$|\xi_\infty(ix)| \geq ce^{c|x|}.$$

Link with the Riemann zeta function

Let ζ be the Riemann zeta function.

- The Riemann hypothesis says that all the zeros of ζ whose real part are in $[0,1]$ are in fact on the critical line $\{s \in \mathbb{C}, \Re(s) = 1/2\}$.
- A conjecture by Montgomery, generalized by Rudnik and Sarnak, says, in a sense which can be made precise, that the distribution of zeros of ζ, properly renormalized, tends to the distribution of a determinantal sine-kernel process, when the imaginary part goes to infinity.
- Using this conjecture and a classical expression of ζ as a Hadamard product, we have stated the following conjecture.

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Conjecture

If U is a uniform variable on $[0,1]$*, then we have the convergences in law:*

$$\left(\frac{\zeta(\frac{1}{2}+iTU-\frac{2i\pi z}{\log T})}{\zeta(\frac{1}{2}+iTU)}\right)_{z\in\mathbb{C}} \underset{T\to\infty}{\longrightarrow} (\xi_\infty(z))_{z\in\mathbb{C}},$$

$$\left(\frac{-2i\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iTU-\frac{2i\pi z}{\log T}\right)\right)_{z\in\mathbb{C}\setminus\mathbb{R}} \underset{T\to\infty}{\longrightarrow} \left(\frac{\xi'_\infty}{\xi_\infty}(z)\right)_{z\in\mathbb{C}\setminus\mathbb{R}}.$$

- For this last convergence in law, we also expect the corresponding convergence for the moments of ratios of ζ and the moments of ζ'/ζ. Such a convergence is directly related to conjectures by Goldston, Gonek and Montgomery.
- It has been proven by Rodgers that under Riemann hypothesis, the conjectures by Montgomery, Rudnik, Sarnak on the distribution of the zeros of ζ imply our conjecture on the moments of ratios of ζ.

Conjecture

If U is a uniform variable on $[0,1]$, then we have the convergences in law:

$$\left(\frac{\zeta(\frac{1}{2} + iTU - \frac{2i\pi z}{\log T})}{\zeta(\frac{1}{2} + iTU)} \right)_{z \in \mathbb{C}} \underset{T \to \infty}{\longrightarrow} (\xi_\infty(z))_{z \in \mathbb{C}},$$

$$\left(\frac{-2i\pi}{\log T} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + iTU - \frac{2i\pi z}{\log T} \right) \right)_{z \in \mathbb{C} \backslash \mathbb{R}} \underset{T \to \infty}{\longrightarrow} \left(\frac{\xi'_\infty}{\xi_\infty}(z) \right)_{z \in \mathbb{C} \backslash \mathbb{R}}.$$

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Thank you for your attention!