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Invariance principle and local limit theorem for the Random Conductance Model

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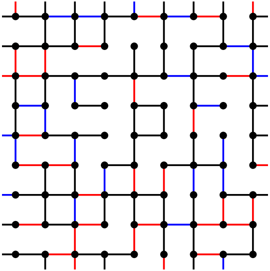
The Random Conductance Model

Intuitive description

- Put random conductances (or weights) $\omega_e \in [0,\infty)$ on the edges of the Euclidean lattice $(\mathbb{Z}^d, E_d)$, $d \geq 2$.
- Look at a continuous time Markov chain X_t with jump probabilities proportional to the edge conductances. Then the jump probability from x to $y \sim x$ is

$$P_{xy} = \frac{\omega_{xy}}{\sum_{z \sim x} \omega_{xz}}.$$



Bond conductivities: blue $\ll 1$, black ≈ 1, red $\gg 1$.

Definitions

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- **Random walk.** For each $\omega \in \Omega$ let P_x^ω be the probability law on $D([0,\infty),\mathbb{Z}^d)$ which makes the coordinate process X_t a Markov chain starting in x with generator

$$\mathcal{L}^\omega f(x) = \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy}(f(y) - f(x))$$

with $\mu_x = \sum_{y \sim x} \omega_{xy}$.

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with $\mu_x = \sum_{y\sim x}\omega_{xy}$.
- **Heat kernel.** Let

$$p_t^\omega(x,y) = \frac{P_x^\omega(X_t = y)}{\mu_y} = p_t^\omega(y,x)$$

be the heat kernel of X (transition density with respect to μ).

Problems

Goal: Understand the long-time-behaviour of the dynamics:

- **Quenched functional CLT (QFCLT)** with diffusivity Σ: Let $X_t^{(n)} = \frac{1}{n} X_{n^2 t}$, and W be a BM($\mathbb{R}^d$). Then, for $\mathbb{P}$-a.a. ω, under P_0^ω,

$$X^{(n)} \Rightarrow \Sigma \cdot W \qquad \text{(with } \Sigma \text{ non-degenerate?)}$$

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- **Quenched local limit theorem**:

$$\lim_{n\to\infty} \sup_{|x|\leq K} \sup_{t\in[T_1,T_2]} \left| n^d p^\omega_{n^2 t}(0, \lfloor nx \rfloor) - \frac{1}{\mathbb{E}[\mu_0]} k_t^\Sigma(0,x) \right| = 0, \qquad \mathbb{P}\text{-a.s.},$$

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- **Gaussian bounds (GB)** on $p_t^\omega(x,y)$: There exist r.v. $N_x(\omega)$ such that

$$p_t^\omega(x,y) \leq c_1 t^{-d/2} e^{-c_2 |x-y|^2/t}, \qquad \text{if } t \geq |x-y| \vee N_x,$$

  and similar lower bounds.

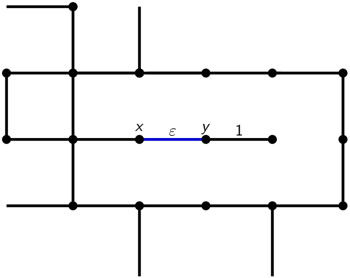
Results in the i.i.d. case: QFCLT

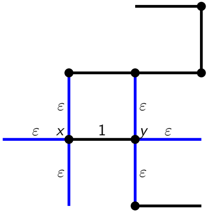
- “Elliptic”: $0 < c_1 \leq \omega_e \leq c_2 < \infty$. Sidoravicius and Sznitman (2004).
- “Percolation”: $\omega_e \in \{0, 1\}$ (and $p_+ = \mathbb{P}[\omega_e > 0] > p_c$.) Sidoravicius and Sznitman (2004), Berger and Biskup (2007), Mathieu and Piatnitski (2007).
- “Bounded above”: $\omega_e \in [0, 1]$ (and $p_+ > p_c$.) QFCLT with $\Sigma = \sigma \, \mathrm{Id} > 0$: Biskup and Prescott (2007), Mathieu (2007).
- “Bounded below”: $\omega_e \in [1, \infty)$. Barlow and Deuschel (2010). Barlow and Černý (2011) 'fractional kinetics motion'.
- “General i.i.d. case“: $\omega_e \geq 0$ (and $p_+ > p_c$.) QFCLT with $\Sigma = \sigma \, \mathrm{Id} \geq 0$ proved by A., Barlow, Deuschel, Hambly (2013). **No moment condition required!**

Results in the i.i.d. case: Local limit theorem

- "Elliptic": $0 < c_1 \leq \omega_e \leq c_2 < \infty$. Barlow and Hambly ('09).
- "Percolation": $\omega_e \in \{0, 1\}$ (and $p_+ > p_c$.) Barlow and Hambly ('09).
- "Bounded above": $\omega_e \in [0, 1]$ (and $p_+ > p_c$.) Berger, Biskup, Hoffmann, Kozma ('08) showed sub-Gaussian heat kernel decay can occur, so **Gaussian bounds and a local limit theorem may fail!**
- Boukhadra, Kumagai, Mathieu ('14): Sharp conditions on the tail of the conductances near 0.

Traps





Invariance Principle for ergodic environments

- Biskup (2011): QFCLT if $\mathbb{E}[\omega_e] < \infty$ and $\mathbb{E}[\omega_e^{-1}] < \infty$ in $d = 2$.
- Barlow, Burdzy and Timár (2013): Example with $\mathbb{E}[\omega_e^p \vee \omega_e^{-p}] < \infty$, $p < 1$, for which the QFCLT fails but the averaged FCLT holds.

Theorem (A., Deuschel, Slowik (2015))

Suppose $d \geq 2$. Let $(\omega_e)_{e \in E_d}$ be stationary ergodic and $p, q \in (1, \infty]$ be such that $1/p + 1/q < 2/d$ and assume that

$$\mathbb{E}\big[(\omega_e)^p\big] < \infty \quad \textit{and} \quad \mathbb{E}\big[(\omega_e)^{-q}\big] < \infty$$

for any $e \in E_d$. Then, QFCLT holds with a deterministic non-degenerate diffusivity matrix Σ.

Quenched local limit theorem for ergodic environments

Theorem (A., Deuschel, Slowik; to appear)

(i) Suppose $d \geq 2$. Let $p, q \in (1, \infty)$ be such that $1/p + 1/q < 2/d$ and assume that $\mathbb{E}[(\omega_e)^p] < \infty$ and $\mathbb{E}[(\omega_e)^{-q}] < \infty$. Then,

$$\lim_{n\to\infty} \sup_{|x|\leq K} \sup_{t\in[T_1,T_2]} \left| p^{\omega}_{n^2 t}(0, \lfloor nx \rfloor) - \frac{1}{\mathbb{E}[\mu_0]} k^{\Sigma}_t(0, x) \right| = 0, \qquad \mathbb{P}\text{-a.s.} \qquad (1)$$

(ii) For $p, q \in (1, \infty)$ such that $1/p + 1/q > 2/d$ there exists an ergodic environment with $\mathbb{E}[(\omega_e)^p] < \infty$ and $\mathbb{E}[(\omega_e)^{-q}] < \infty$, under which (1) does not hold.

Proof of (i):

- QFCLT
- Hölder continuity of the heat kernel

Proof of (ii): Definition of the environment

- Let $(\xi_e)_{e \in E_d}$ be i.i.d. $U([0,1])$ and for all k large enough

$$\theta_e^k := \begin{cases} 1 & \text{if } \xi_e \leq 2^{-k} < p_c, \\ 0 & \text{otherwise.} \end{cases}$$

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- $\mathcal{O}^k := \{e \in E_d : \theta_e^k = 1\}$ and let $\{\mathcal{H}_j^k\}_{j \geq 1}$ be the connected components of $(\mathbb{Z}^d, \mathcal{O}^k)$.

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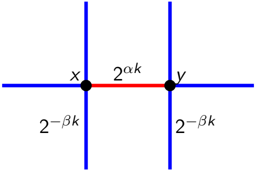
- Initialise $\omega_e = 1$ for all e. At the k-th iteration step pick one edge $\{x, y\}$ in every $\mathcal{H}_j^k$ uniformly at random and put a k-trap on it:



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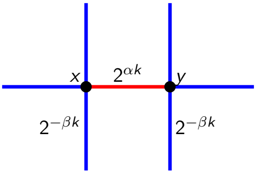

- In particular, for every k-trap $\{x, y\}$,

$$P_x^\omega[X_{2^{(\alpha+\beta)k}} = x] \geq C_1 > 0.$$

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- Let $\alpha < 1/p$ and $\beta < 1/q$. Since

$$\min_{e' \sim e} \xi_{e'}^{\beta} \;\le\; \min_{e' \sim e} \min_{k : \xi_{e'} \le 2^{-k}} 2^{-\beta k} \;\le\; \omega_e \le \max_{k : \xi_e \le 2^{-k}} 2^{\alpha k} \;\le\; \xi_e^{-\alpha},$$

we have

$$\mathbb{E}\big[\omega_e^p\big] \;\le\; \mathbb{E}\big[\xi_e^{-\alpha p}\big] \;=\; \int_0^1 u^{-\alpha p}\, du \;<\; \infty,$$

$$\mathbb{E}\big[\omega_e^{-q}\big] \;\le\; \mathbb{E}\big[\max_{e' \sim e} \xi_{e'}^{-\beta q}\big] \;\le\; 2d \int_0^1 u^{-\beta q}\, du \;<\; \infty,$$

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- A Borel-Cantelli argument shows that for all n large enough we find an n-trap $\bar{e}_n$ in $B(0, R_n)$ with $R_n = (n 2^n)^{1/d}$.

Disproof of the local limit theorem

- Since $1/p + 1/q > 2/d$ there exist $\alpha < 1/p$ and $\beta < 1/q$ such that $(\alpha+\beta)d/2 > 1$
- Suppose that the local limit theorem holds. Then, for $\mathbb{P}$-a.e. ω for all $t \geq N_0(\omega)$ and $x \in B(0, \sqrt{t})$,

$$C_2 t^{-d/2} \leq p_t^\omega(0,x) \leq C_3 t^{-d/2}.$$

- Setting $t_1 := R_n^2$ and $t_2 := t_1^{(\alpha+\beta)d/2}$, we have for any $x \in B(0, R_n)$,

$$\frac{P_0^\omega[X_{t_1+t_2} = x]}{P_0^\omega[X_{t_1} = x]} = \frac{p_{t_1+t_2}^\omega(0,x)}{p_{t_1}^\omega(0,x)} \leq c\left(1 + \frac{t_2}{t_1}\right)^{-\frac{d}{2}} \to 0 \qquad \text{as } n \to \infty$$

  since $t_2/t_1 = R_n^{(\alpha+\beta)d-2} \to \infty$.
- On the other hand, using the Markov property,

$$\frac{P_0^\omega[X_{t_1+t_2} = \bar{e}_n]}{P_0^\omega[X_{t_1} = \bar{e}_n]} \geq P_0^\omega\left[X_{t_1+t_2} = \bar{e}_n \,\middle|\, X_{t_1} = \bar{e}_n\right] = P_{\bar{e}_n}^\omega[X_{t_2} = \bar{e}_n] \geq C_1.$$

Upper Gaussian estimates for ergodic environments

Theorem (A., Deuschel, Slowik; arXiv 2014)

Let $d \geq 2$ and $p, q \in (1, \infty)$ be such that

$$1/p + 1/q < 2/d$$

and assume that $\mathbb{E}\big[(\omega_e)^p\big] < \infty$ and $\mathbb{E}\big[(\omega_e)^{-q}\big] < \infty$. Further let $N_x = N_x(\omega)$ be such that

$$\sup_{n \geq N_x} \|\mu\|_{p,B(x,n)}^p \leq 2\,\mathbb{E}\big[\mu_0^p\big], \qquad \sup_{n \geq N_x} \|\nu\|_{q,B(x,n)}^q \leq 2\,\mathbb{E}\big[\nu_0^q\big].$$

Then, there exist constants $c_i > 0$ such that for any t and x with $\sqrt{t} \geq N_x$ and all $y \in \mathbb{Z}^d$,

$$p_t^\omega(x,y) \leq c_1\, t^{-d/2} \exp\big(-c_2 |x-y|^2/t\big), \qquad \text{if } t \geq |x-y|.$$

Summary and Outlook

What has been acheived so far:

- QFCLT and local limit theorem for ergodic environment.
- Elliptic and parabolic Harnack inequalities on graphs beyond strong ellipticity (not shown in this talk).

Next tasks and major challenges:

- Show that $p = q = 1$ is optimal for QFCLT.
- Quantititive QFLT, Berry-Esseen-Theorem.
- Ergodic environment on random graphs.
- Dynamic random conductance model
  - QFCLT
  - quenched local CLT

  (proved by A. (2012) for the elliptic case under mixing assumptions)