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Convergence of two-dimensional random walk loop soup clusters to CLE

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Structure of the talk

1. Introduction of the loop soup model

2. Critical intensity on the discrete half-plane

3. Scaling limit of clusters on the discrete half-plane

Definition of the loop soups

Infinite measure on loops associated to a wide range of symmetric Markov processes.

S locally compact second countable space, Borel σ-algebra.

$(X_t)_{0\leq t<\zeta}$ cadlag Feller process on S, killing time $\zeta \in (0,+\infty]$.

Transition densities $p_t(x,y)$ relative to a σ-finite m on S, continuous in (t,x,y) and symmetric.

Bridge probabilities $\mathbb{P}^t_{x,y}$ continuous in (t,x,y).

Measure on loops

$$\mu(\cdot) := \int_{x\in S}\int_{t>0} \mathbb{P}^t_{x,x}(\cdot) p_t(x,x)\frac{dt}{t}m(dx)$$

$\alpha > 0$. **Loop soup** $\mathcal{L}_\alpha$: Poisson point process of intensity $\alpha\mu$.

Clusters of Brownian loops in dimension 2

D open simply connected proper sub-domain of $\mathbb{C}$. Brownian motion in D killed at hitting ∂D. Brownian loop soup $\mathcal{L}_\alpha$ in D.

Law of $\mathcal{L}_\alpha$ invariant under conformal transformation of D (up to a time change).

Clusters of $\mathcal{L}_\alpha$: $\gamma, \gamma' \in \mathcal{L}_\alpha$ are in the same cluster if joint by a chain of consecutively intersecting loops of $\mathcal{L}_\alpha$.

Sheffield,Werner(2010):

- If $\alpha > \frac{1}{2}$, a single cluster, everywhere dense in D.
- If $\alpha \in (0, \frac{1}{2}]$, infinitely many clusters. The outer boundaries of outermost clusters are a Conformal Loop Ensemble CLE_κ

$$2\alpha = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

Clusters of Brownian loops in dimension 2 (picture)

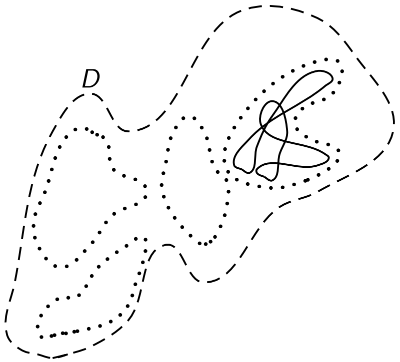

Fig. 1: Representation of four CLE_κ loops (dotted lines) and Brownian loops (full lines) inside one of it.

Random walk loop soups

$\mathcal{G} = (V, E)$ connected undirected graph.
V at most countable.
Degree of vertices finite.
Conductances $C(x, y) > 0$ on edges $\{x, y\}$.
Possibly a killing measure $k(x) > 0$ on V.

Markov jump process $(X_t)_{t\geq 0}$ on $\mathcal{G}$:

- Transition rate from x to a neighbour y: $C(x, y)$.
- Killing rate at x: $k(x)$.

Random walk loop soups associated to $(X_t)_{t\geq 0}$.

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- Answer: For $\alpha \in (0, \frac{1}{2}]$ the scaling limit is a CLE_κ (L.2015).

Le Jan's isomorphism

Framework: $\mathcal{G} = (V, E)$ undirected connected graph. Conductances $C(x, y) > 0$ on edges. Possibly a killing measure $k(x)$.
Assumption: The Markov jump process is **transient**.

Occupation field of $\mathcal{L}_\alpha$:

$$\widehat{\mathcal{L}}_\alpha^x := \sum_{\gamma \in \mathcal{L}_\alpha} \int_0^{t_\gamma} 1(\gamma(s) = x) ds < +\infty, \quad x \in V$$

Gaussian free field ϕ on $\mathcal{G}$: centred Gaussian field whose covariance fonction is the Green function of the Markov jump process.

Le Jan's isomorphism (2009):

$$(\widehat{\mathcal{L}}_{\frac{1}{2}}^x)_{x \in V} \overset{(d)}{=} (\frac{1}{2} \phi_x^2)_{x \in V}$$

Sign of ϕ?

Recovering the sign of the GFF

Framework: $\mathcal{G} = (V, E)$ undirected connected graph. Conductances $C(x, y) > 0$ on edges. Possibly killing measure $k(x)$. Transient Markov jump process. Random walk loop soup $\mathcal{L}_\alpha$. Gaussian free field ϕ.

Theorem (L. 2014)

There is an (explicit) coupling between $\mathcal{L}_{\frac{1}{2}}$ and ϕ such that:

(i) $\forall x \in V, \widehat{\mathcal{L}}^x_{\frac{1}{2}} = \frac{1}{2}\phi_x^2$ *(Le Jan's isomorphism).*

(ii) *The sign of ϕ is constant on each cluster of $\mathcal{L}_{\frac{1}{2}}$.*

Corollary

On the discrete half-plane $\mathbb{H}$ the random walk loop soup $\mathcal{L}_{\frac{1}{2}}$ has no infinite cluster. The critical intensity is $\frac{1}{2}$.

Description of the coupling

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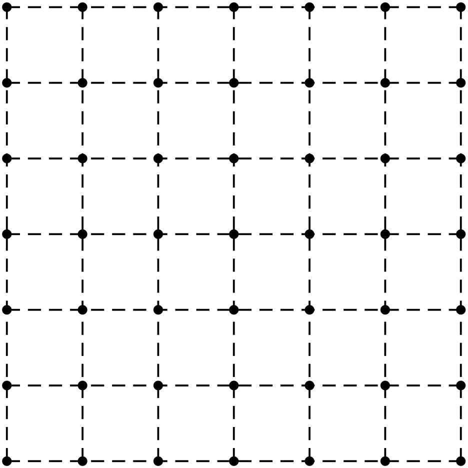

Fig. 2: Construction of sign(ϕ).

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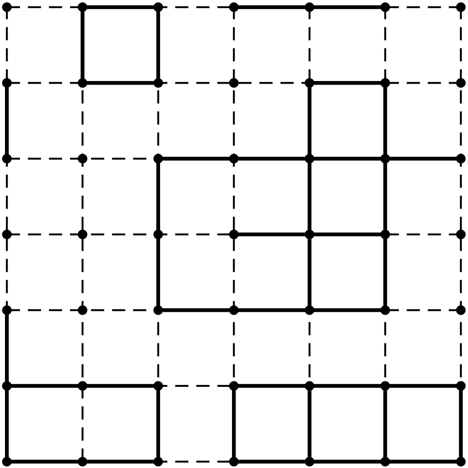

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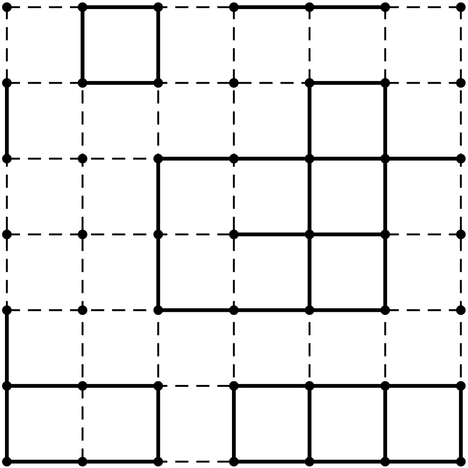

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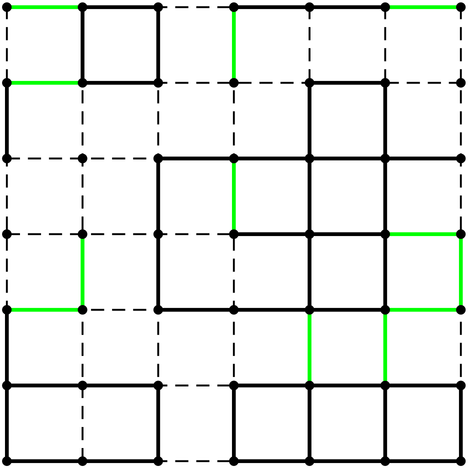

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(iii) Open each edge $\{x, y\}$ non-visited by a loop cond. indep. with probability

$$1 - e^{-2(\widehat{\mathcal{L}}^x_{\frac{1}{2}} \widehat{\mathcal{L}}^y_{\frac{1}{2}})^{\frac{1}{2}}}$$

$\to$ larger clusters.

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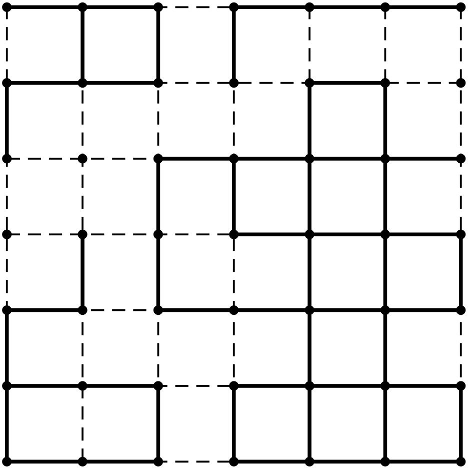

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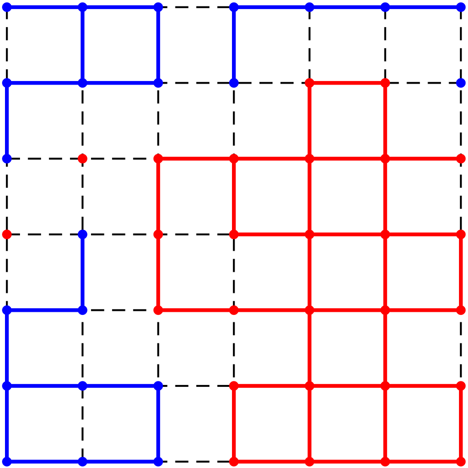

Fig. 2: Construction of $\text{sign}(\phi)$.

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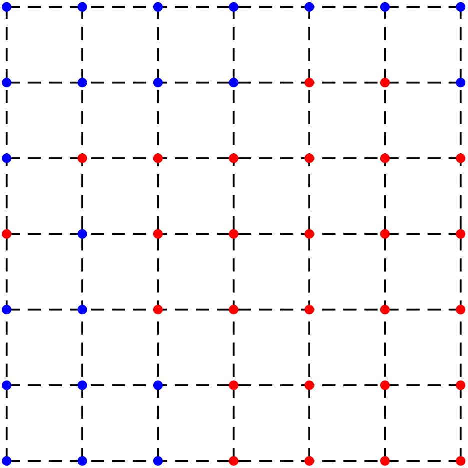

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Use of metric graphs

Idea: introduce the metric graph $\widetilde{\mathcal{G}}$ associated to $\mathcal{G}$.
Edges replaced by lines of length $\rho(x,y) = \frac{1}{2}C(x,y)^{-1}$.
Brownian motion $\widetilde{B}^{\widetilde{\mathcal{G}}}$ on $\widetilde{\mathcal{G}}$ killed by the killing measure $\tilde{k} := \sum_{x \in V} k(x)\delta_x$.
The trace of $\widetilde{B}^{\widetilde{\mathcal{G}}}$ on the vertices is the Markov jump process on $\mathcal{G}$. Time spent at $x \in V$ by the Markov jump process $=$ local time at x of $\widetilde{B}^{\widetilde{\mathcal{G}}}$.

GFF $\tilde{\phi}$ on $\widetilde{\mathcal{G}}$ $=$ GFF ϕ on $\mathcal{G}$ $+$ Brownian bridges.
$\widetilde{\mathcal{L}}_\alpha$ loop soup associated to $\widetilde{B}^{\widetilde{\mathcal{G}}}$.
$\alpha = \frac{1}{2}$: Clusters of $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ $=$ connected components of $\{\tilde{\phi}^2 > 0\}$
$=$ sign components of $\tilde{\phi}$.

Random walk loops $\mathcal{L}_\alpha$ $=$ trace of $\widetilde{\mathcal{L}}_\alpha$ on V.
Clusters of $\widetilde{\mathcal{L}}_\alpha$ $=$ clusters of $\mathcal{L}_\alpha$ $+$ open additional edges (excursions inside the edges and loops that do not visit V).

Important identities

G Green function of the Markov jump process on $\mathcal{G}$.
$x, y \in V$. Equivalent conductance between x and y:

$$C^{eq}(x,y) = \frac{G(x,y)}{(G(x,x)G(y,y) - G(x,y)^2)^2}$$

$$\mathbb{P}\left(x \overset{\text{cluster of } \widetilde{\mathcal{L}}_{\frac{1}{2}}}{\longleftrightarrow} y\right) = \mathbb{E}\left[\operatorname{sign}(\phi_x)\operatorname{sign}(\phi_y)\right] = \frac{2}{\pi}\arcsin\left(\frac{G(x,y)}{\sqrt{G(x,x)G(y,y)}}\right)$$

$$\mathbb{P}\left(x \overset{\text{cluster of } \widetilde{\mathcal{L}}_{\frac{1}{2}}}{\longleftrightarrow} y \,\middle|\, \widehat{\mathcal{L}}^{x}_{\frac{1}{2}} = u, \widehat{\mathcal{L}}^{y}_{\frac{1}{2}} = v, \nexists\gamma \in \widetilde{\mathcal{L}}_{\frac{1}{2}}\ x, y \in \gamma\right) = 1 - e^{-2C^{eq}(x,y)\sqrt{uv}}$$

Presentation of convergence results

$\mathbb{H} = \{\Im(z) > 0\}$. Brownian motion on $\mathbb{H}$ killed on $\{\Im(z) = 0\}$. $\alpha \in (0, \frac{1}{2}]$: outermost boundaries of clusters of loops $= CLE_\kappa$.

$\mathtt{H}_n = (n^{-1}\mathbb{Z}) \times (n^{-1}\mathbb{N})$. Conductances $= \frac{n}{2}$. Markov jump process on $(n^{-1}\mathbb{Z}) \times \{0\}$. Loops $\mathcal{L}_{n,\alpha}$. Outermost boundaries of clusters of loops: $\mathcal{F}_{ext}(\mathcal{L}_{n,\alpha})$.

$\widetilde{\mathtt{H}}_n$ metric graph associated to $\mathtt{H}_n$. Edge length $= n^{-1}$. Brownian motion on $\widetilde{\mathtt{H}}_n$. Loops $\widetilde{\mathcal{L}}_{n,\alpha}$. Outermost boundaries of clusters of loops: $\mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\alpha})$.

Brug, Camia, Lis (2014): $\theta \in (\frac{16}{9}, 2)$. $\alpha \in (0, \frac{1}{2}]$. $\mathcal{L}^{n^\theta}_{n,\alpha}$ loops in $\mathcal{L}_{n,\alpha}$ that perform at least n^θ jumps. $\mathcal{F}_{ext}(\mathcal{L}^{n^\theta}_{n,\alpha})$ converges in law to CLE_κ.

Theorem (L. 2015)

$\alpha \in (0, \frac{1}{2}]$. $\mathcal{F}_{ext}(\mathcal{L}_{n,\alpha})$ and $\mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\alpha})$ converge in law to CLE_κ.

Boundaries of clusters on metric graph (picture)

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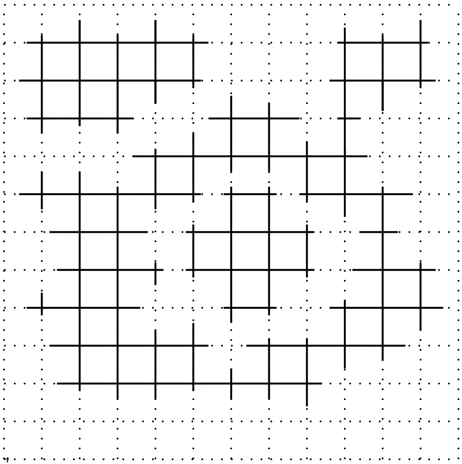

Fig. 3: Three clusters (thin full lines) of $\widetilde{\mathcal{L}}_{n,\alpha}$, two external and one surrounded. In thick lines the elements of $\mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\alpha})$.

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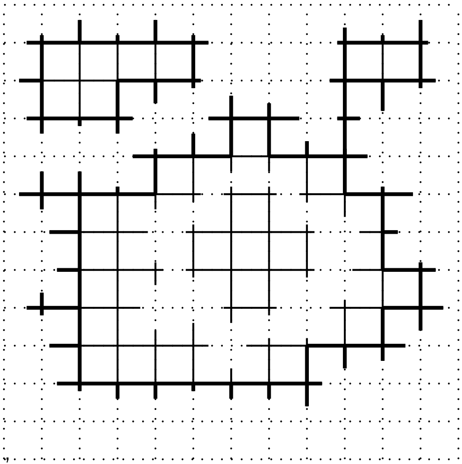

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In thick lines the elements of $\mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\alpha})$.

Sketch of the proof of convergence

Suffices to prove the convergence on the metric graph: $\mathcal{F}_{ext}(\mathcal{L}_{n,\alpha})$ lies between $\mathcal{F}_{ext}(\mathcal{L}^{n^\theta}_{n,\alpha})$ and $\mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\alpha})$.

The "limit" of $\mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\alpha})$ is "at least as large as" CLE_κ (Brug, Camia, Lis).

Suffices to prove the convergence for $\alpha = \frac{1}{2}$: If "$\lim \mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\alpha})$" is "strictly larger" than CLE_κ for an $\alpha < \frac{1}{2}$ then this is also the case for $\alpha = \frac{1}{2}$.

One just needs to "bound from above" $\mathcal{F}_{ext}(\widetilde{\mathcal{L}}_{n,\frac{1}{2}})$ by CLE_4.

"Upper bound"

Two independent Poisson families of excursions on $\widetilde{\mathbb{H}}_n$.
From $(n^{-1}]\!] -\infty, 0]\!]) \times \{0\}$ to itself: $\widetilde{\mathcal{E}}_{n,u}$.
From $(n^{-1}[\![n, \lfloor nq \rfloor]\!]) \times \{0\}$ to itself: $\widetilde{\mathcal{E}}_{n,v}(q)$, $q > 1$.
$u, v > 0$ intensity parameter.

Two independent Poisson families of Brownian excursions on $\mathbb{H}$.
$\mathcal{E}_u$: from and to $(-\infty, 0) \times \{0\}$. Scaling limit of $\widetilde{\mathcal{E}}_{n,u}$.
$\mathcal{E}_v(q)$: from and to $(1, q) \times \{0\}$. Scaling limit of $\widetilde{\mathcal{E}}_{n,v}(q)$.

Event: Either an excursion from $\widetilde{\mathcal{E}}_{n,u}$ intersects an excursion from $\widetilde{\mathcal{E}}_{n,v}(q)$ or one of each intersects the same cluster of $\widetilde{\mathcal{L}}_{n,\frac{1}{2}}$.
Probability: $p^n_{u,v}(q)$.

Event: Either an excursion from $\mathcal{E}_u$ intersects an excursion from $\mathcal{E}_v(q)$ or one of each intersects the same CLE_4 loop.
Probability: $p_{u,v}(q)$.

If for some $u, v > 0$ and $q > 1$, $\lim p^n_{u,v}(q) = p_{u,v}(q)$ then we have an "upper bound".

"Bounding from above" (picture)

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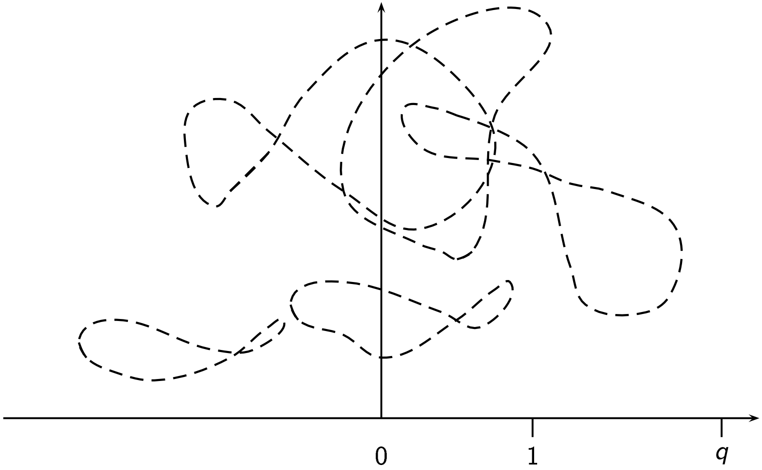

Fig. 4: Two excursions (full lines) conneced by a chain of two loops (dashed lines).

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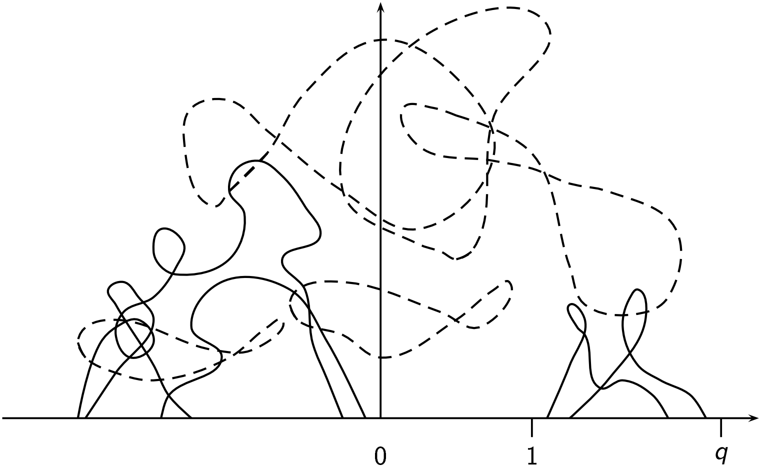

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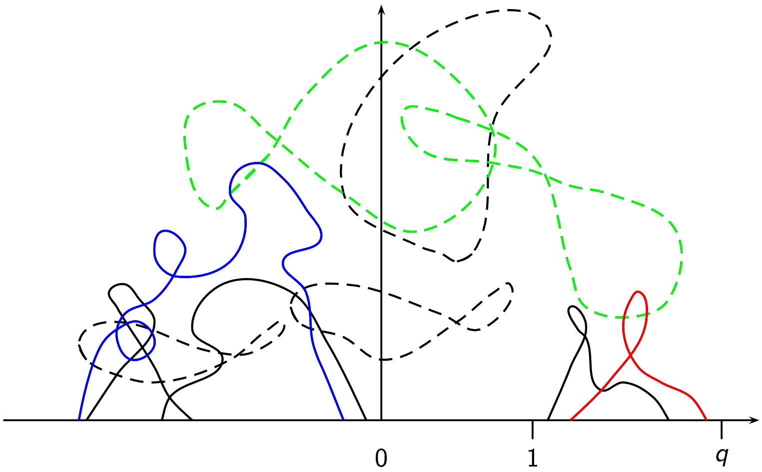

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Computation of $p^n_{u,v}(q)$

$p^n_{u,v}(q)$ and $\lim_{n\to+\infty} p^n_{u,v}(q)$ explicit.

Application of the duality with the GFF on a metric graph.

$(n^{-1}]\!] -\infty, 0]\!]) \times \{0\}$ identified to one vertex $\triangleleft_n$.
$\widetilde{\mathcal{E}}_{n,u}$: loops visiting $\triangleleft_n$.

$(n^{-1}[\![n, \lfloor nq \rfloor]\!]) \times \{0\}$ identified to one vertex $\triangleright_n(q)$.
$\widetilde{\mathcal{E}}_{n,v}(q)$: loops visiting $\triangleright_n(q)$.

$C^{eq}_n(q)$ equivalent conductance between $\triangleleft_n$ and $\triangleright_n(q)$.

$$p^n_{u,v}(q) = 1 - e^{-2C^{eq}_n(q)\frac{8\pi\sqrt{uv}}{n}}$$

$$\frac{1}{n} C^{eq}_n(q) = \frac{1}{8\pi} \log(q) + O\left(\frac{1}{n}\right)$$

$$\lim_{n\to+\infty} p^n_{u,v}(q) = 1 - q^{-2\sqrt{uv}}$$

Computation of $p_{u,v}(q)$

If $u = \frac{1}{4}$ and v arbitrary one can establish a differential equation in q for $1 - p_{\frac{1}{4},v}(q)$.

Loops of CLE_4 intersecting an excursion from $\mathcal{E}_{\frac{1}{4}}$: The right boundary is a chordal SLE_4 curve from 0 to ∞.

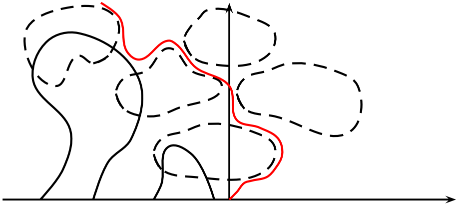

Fig. 5: Right boundary (in red) of CLE_4 loops intersecting $\mathcal{E}_{\frac{1}{4}}$.

$p_{\frac{1}{4},v}(q)$ = probability that an excursion from $\mathcal{E}_v(q)$ intersects an independent SLE_4.

Differential equation for $1 - p_{\frac{1}{4},v}(q)$: $f'' + \frac{1}{q} f' - \frac{v}{q^2} f = 0 \quad (E)$

$\lim_{n \to +\infty} (1 - p^n_{\frac{1}{4},v}(q)) = q^{-\sqrt{v}}$ satisfies (E).

Thank you for your attention!