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Interacting Browninan motions in infinite dimensions and random matrix theory

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The 38th Conf on Stochastic Processes and Their Applications
2015/7/14/Tue, Oxford

- Random matrices /Airy RPFs
- Dynamical soft edge scaling / Infinite-dim SDEs of Airy RPFs
- Sine RPFs (Dyson's medel), Bessel RPFs
- General theorems of ISDEs on strong solutions:
- Gibbs measures: Lennard-Jones 6-12, Riesz potentials
- Ginibre RPF (a strict Coulomb RPF)
- Dynamical rigidity of 2D Coulomb interacting Brownian motions
- Beyond 2 body potentials: planar GAF

- The Gaussian unitary ensembles are Hermitian random matrices

$$M^N = \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \cdots & \cdots & \\ m_{N1} & \cdots & m_{NN} \end{pmatrix} \quad m_{ij} = \bar{m}_{ji}.$$

Here $m_{ij} = m_{ij}^1 + \sqrt{-1} m_{ij}^2$. $\{m_{ij}^1, m_{ij}^2, m_{kk}\}_{1 \le i \le j \le N, 1 \le k \le N}$ are independent real Gaussian r. v. with mean free, variance 1.

- The distribution of eigen values of M^N is given by ($\beta = 2$)

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \tag{1}$$

- The dist of $N^{-1} \sum_{i=1}^N \delta_{x_i/\sqrt{N}}$ under m_β^N converges to

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4 - x^2}dx \qquad \text{(the semi-circle law)}\ . \tag{2}$$

Soft edge and bulk scalings (2/2)

:1b

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4-x^2}dx \qquad \text{(the semi-circle law)} . \qquad (3)$$

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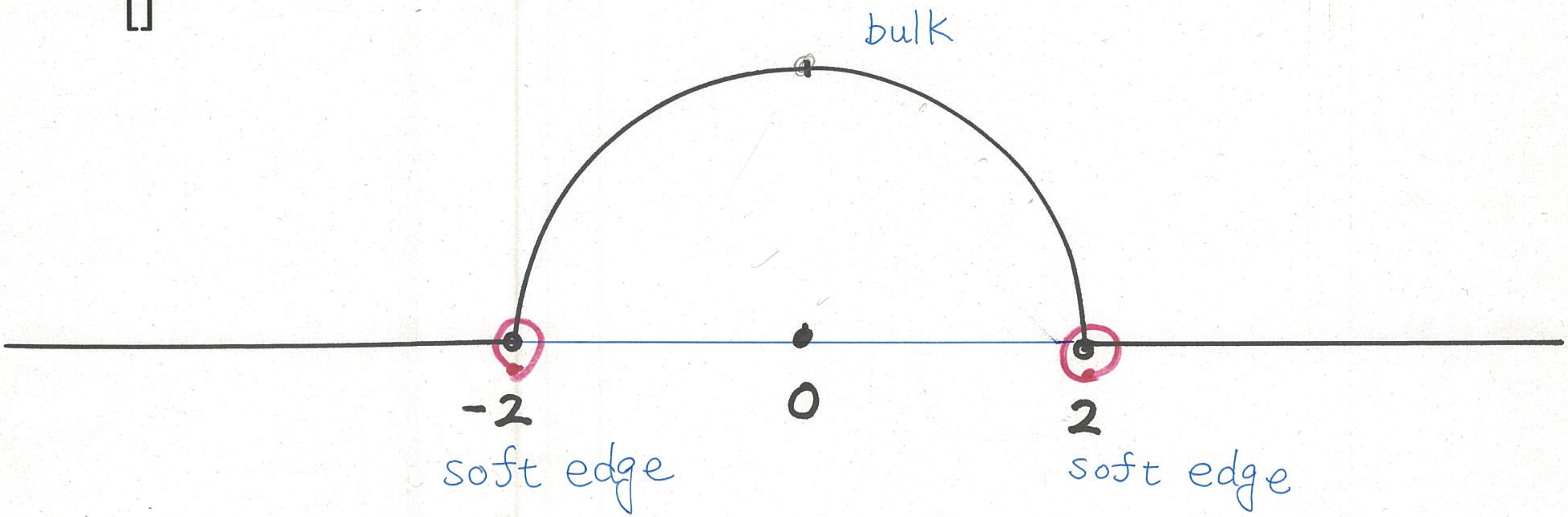

Soft edge at 2 $\qquad x \longmapsto 2\sqrt{N} + \dfrac{s}{N^{1/6}}$

Bulk at 0 $\qquad x \longmapsto \dfrac{s}{\sqrt{N}}$

Airy RPF: $\mu_{\mathsf{Ai},\beta}$

- In $m_\beta^N(d\mathbf{x}_N)$, take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$. Then we have

$$\check{\mu}^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{k=1}^N |2\sqrt{N}+N^{-1/6}s_k|^2} d\mathbf{s}_N.$$

- Let $\mu^N_{\mathsf{Ai},\beta} = \check{\mu}^N_{\mathsf{Ai},\beta} \circ \mathfrak{u}^{-1}$. $\quad (\mathfrak{u}(\mathbf{s}) = \sum_i \delta_{s_i}$ is the unlabel map)

Then $\mu_{\mathsf{Ai},\beta}$ is the TDL of $\mu^N_{\mathsf{Ai},\beta}$:

$$\lim_{N\to\infty} \mu^N_{\mathsf{Ai},\beta} = \mu_{\mathsf{Ai},\beta} \qquad \text{(soft-edge scaling)}$$

- $\beta = 2 \Rightarrow \mu_{\mathsf{Ai},2}$ is the det RPF generated by (K_{Ai}, dx):

$$\rho^m_{\mathsf{Ai},2}(\mathbf{x_m}) = \det[K_{\mathsf{Ai}}(x_i, x_j)]_{1\le i,j\le m}$$

Here $\rho^m_{\mathsf{Ai},2}$ are correlation funs, and K_{Ai} is the Airy kernel:

$$K_{\mathsf{Ai}}(x,y) = \frac{\mathsf{Ai}(x)\mathsf{Ai}'(y) - \mathsf{Ai}'(x)\mathsf{Ai}(y)}{x-y} \tag{4}$$

We consider the natural SDE associated with

$$\breve{\mu}^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N} |s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N. \tag{5}$$

We consider the energy form associated with (5). Then

$$\int_{\mathbb{R}^N} \frac{1}{2}\sum_{i=1}^N \frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_i}\, d\breve{\mu}^N_{\mathsf{Ai},\beta} =$$

$$\int_{\mathbb{R}^N} \frac{1}{2}\{-\Delta f - \sum_{i=1}^N \frac{\partial f}{\partial s_i}(\sum_{j\neq i} \frac{\beta}{s_i - s_j} - \beta\{N^{\frac{1}{3}} + \frac{s_i}{2N^{\frac{1}{3}}}\})\} g\, d\breve{\mu}^N_{\mathsf{Ai},\beta}$$

Hence we obtain the SDE of the natural N particle dynamics:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}(\{\sum_{j=1,j\neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}}\} - \{N^{\frac{1}{3}} + \frac{1}{2N^{\frac{1}{3}}}X_t^{N,i}\})dt \tag{6}$$

We will detect the limit ISDE of (6) and solve it.

Let $\varrho^N(x) = N^{\frac{1}{3}} \varsigma(xN^{-\frac{2}{3}} + 2)$ be the rescaled semicircle. Then

$$\lim_{N\to\infty} \varrho^N(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x) =: \varrho(x)$$

$$N^{\frac{1}{3}} = \int_{\mathbb{R}} \frac{\varrho^N(x)}{-x} dx.$$

Hence we have, as $N \to \infty$,

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\{(\sum_{j=1,j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}}) - (N^{\frac{1}{3}} + \frac{1}{2N^{\frac{1}{3}}} X_t^{N,i})\}dt$$

$$\sim dB_t^i + \frac{\beta}{2}\{(\sum_{j=1,j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}}) - (\int_{\mathbb{R}} \frac{\varrho^N(x)}{-x} dx)\}dt$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{(\sum_{|X_t^j|<r, j\neq i} \frac{1}{X_t^i - X_t^j}) - (\int_{|x|\leq r} \frac{\varrho(x)}{-x} dx)\}dt$$

Let ℓ be a label, i.e. $\ell : \sum_i \delta_{s_i} \longmapsto \mathbf{s} = (s_i) \in \mathbb{R}^\infty$.
Let $\mu^{\ell}_{\mathrm{Ai},\beta} = \mu_{\mathrm{Ai},\beta} \circ \ell^{-1}$.
Thm 1 (O.-Tanemura). *Let* $\beta = 1, 2, 4$.
Consider the ISDE of $\mathbf{X}_t = (X^i_t)_{i\in\mathbb{N}} \in \mathbb{R}^\infty$: $\mathbf{X}_0 = \mathbf{s}$.

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{\substack{|X^j_t|<r, \\ j\neq i}} \frac{1}{X^i_t - X^j_t}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt. \tag{7}$$

(1) (7) *has a pathwise unique, strong solution for* $\mu^{\ell}_{\mathrm{Ai},\beta}$*-a.s.* $\mathbf{s}$.
(2) (7) *has a strong uniqueness in the sense that any solution* $(\mathbf{X}, \mathbf{B})$ *becomes a pathwise unique, strong solution.*
(3) *Let* $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X^i_t}$ *be the associated unlabeled dynamics.*
Then X_t *is a* $\mu_{\mathrm{Ai},\beta}$*-reversible diffusion.*

- Uniqueness holds under the "absolutely continuity condition".

- $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N \in \mathbb{R}^N$ (labeled N particle dynamics):

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \{N^{\frac{1}{3}} + \frac{1}{2N^{\frac{1}{3}}} X_t^{N,i}\} dt$$

- $\mathbf{X}_t = (X_t^i)_{i=1}^\infty \in \mathbb{R}^\infty$ (labeled ∞ particle dynamics):

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{|X_t^j|<r,\, j\neq i} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Take a label of $\mathbf{X}^N$ and $\mathbf{X}$ in decreasing order: $X_t^i > X_t^{i+1}$ for all i. Denote by $\mathbf{X}^{N,m}$ and $\mathbf{X}^m$ the first m-components of $\mathbf{X}^N$ and $\mathbf{X}$.

Thm 2 (O.-Tanemura, Kawamoto-O.). *Let $m \in \mathbb{N}$. Suppose $\mathfrak{u}(\mathbf{X}_0^N) \sim \mu_{\mathrm{Ai},\beta}^N$ and $\mathfrak{u}(\mathbf{X}_0) \sim \mu_{\mathrm{Ai},\beta}$.*

(1) $\beta = 2 \Rightarrow \mathbf{X}^{N,m}$ converge to $\mathbf{X}^m$ weakly in $C([0,\infty);\mathbb{R}^m)$.

(2) $\beta = 1, 4 \Rightarrow$ the limit points are solutions of (7)

If $\beta = 2$, then there exists an algebraic construction.

space-time correlation functions (Sine、Airy、Bessel RPF with $\beta = 2$)

- Multi-time moment generating functions of S-valued X_t:

$$\Psi^{\mathbf{t}}[\mathbf{f}] \equiv \mathbb{E}\left[\exp\left\{\sum_{m=1}^{M} \int_{\mathbb{R}} f_m d\mathsf{X}_{t_m}\right\}\right], \tag{8}$$

Let $\mathbb{K}(s, x; t, y)$ be a extended kernel function.

One can define X_t by $\Psi^{\mathbf{t}}[\mathbf{f}]$.

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \underset{\substack{(s,t)\in\{t_1,t_2,\dots,t_M\}^2,\\ (x,y)\in\mathbb{R}^2}}{\mathrm{Det}} \Big[\delta_{st}\delta(x-y) + \mathbb{K}(s, x; t, y)\chi_t(y)\Big], \tag{9}$$

Here $M \in \mathbb{N}$, $\mathbf{f} = (f_1, f_2, \dots, f_M) \in C_0(\mathbb{R})^M$, $\mathbf{t} = (t_1, t_2, \dots, t_M)$ $(0 < t_1 < \cdots < t_M < \infty)$, $\chi_{t_m} = e^{f_m} - 1$, $1 \le m \le M$。

- Extended Airy kernel:

$$\mathbb{K}_{\mathsf{Ai}}(s,x;t,y) \equiv \begin{cases} \displaystyle\int_0^{\infty} du\, e^{-u(t-s)/2} \mathsf{Ai}(u+x)\mathsf{Ai}(u+y) & \text{if } s<t, \\ K_{\mathsf{Ai}}(x,y) & \text{if } s=t, \\ \displaystyle -\int_{-\infty}^{0} du\, e^{-u(t-s)/2} \mathsf{Ai}(u+x)\mathsf{Ai}(u+y) & \text{if } s>t. \end{cases}$$

- $\mathbb{K}_{\mathsf{Ai}}(s,x;t,y)$ defines the $\mu_{\mathsf{Ai},2}$-reversible stochastic dynamics. (Prähofer-Spohn, Johansson, Katori-Tanemura, and others).

Thm 3 (O.-Tanemura)**.** *Stochastic dynamics given by $\mathbb{K}_{\mathsf{Ai}}(s,x;t,y)$ is same as solutions of Airy$_2$ ISDE.*

- Uniqueness of solution of ISDE are crucial for the proof of Thm 3.
- From algebraic construction we obtain quantative information.
- From analytic construction we obtain qualitative information such as semimartingale property, non-collision property, Ito formula, and so on.

Thm 4.

(1) *Let* $\beta = 1, 2, 4$. *We label* $X^1 > X^2 > \dots$. *Suppose* $\mathsf{X}_0 \sim \mu_{\mathsf{Ai},\beta}$. *Then the distribution of the top particle* X^1 *satisfy*

$$X^1_t \sim F_\beta$$

Here F_β *is the* β *Tracy-Widom distribution* ($\beta = 1, 2, 4$).
X^1_t *equals the Airy process* $\mathcal{A}(t)$ *introduced by Prähofer-Spohn.*
(2) X^1_t *are semi-martingale.*
(3) *For each* $(X^1_t, \dots, X^m_t)$, *Girsanov type formula holds.*

- Johansson'conjecture:

$$H(t) = \mathcal{A}(t) - t^2 \text{ has a unique maximum on } [-T, T]$$

is immediately solved by Thm 4.

THis was solved by Corwin-Hammond, Hägg by different methods.

Sine$_\beta$ RPF: $d = 1$, $\beta = 1, 2, 4$

- In $m_\beta^N(d\mathbf{x}_N)$, take the scaling $x_i \mapsto s_i/\sqrt{N}$:

$$\mu_{\mathrm{sin},\beta}^N(ds) = \frac{1}{\mathcal{Z}_N}\{\prod_{i<j}^N |s_i - s_j|\}^\beta e^{-\frac{\beta}{4N}\sum_{k=1}^N |s_k|^2}\Lambda_N(ds)$$

$$\mu_{\mathrm{sin},\beta} = \lim_{N\to\infty}\mu_{\mathrm{sin},\beta}^N \quad \text{(bulk scaling)}.$$

- Stochastic dynamics are given by

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}}dt - \frac{\beta}{4N}X_t^{N,i}dt \qquad \text{(N-Dyson)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{j\neq i, |X_t^j|<r}^\infty \frac{1}{X_t^i - X_t^j}dt \qquad (\infty\text{-Dyson})$$

- ISDE (∞-Dyson) have unique str sols, $\mathbf{X}_t^{N,m}$ converge $\mathbf{X}_t^m$.

Examples: Sine$_\beta$ RPF (Dyson's model)–bulk scaling limit

- $\beta = 2 \Rightarrow \mu_{\mathrm{sin},\beta}$ is the determinantal RPF generated by (K_{sin}, dx):

$$K_{\mathrm{sin}}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$$

- Extended sine kernel:

$$\mathbb{K}_{\mathrm{sin}}(s,x;t,y) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^1 du\, e^{u^2(t-s)/2} \cos\{u(y-x)\} & \text{if } s<t, \\ K_{\mathrm{sin}}(x,y) & \text{if } s=t, \\ -\dfrac{1}{\pi}\displaystyle\int_1^\infty du\, e^{u^2(t-s)/2} \cos\{u(y-x)\} & \text{if } s>t. \end{cases}$$

gives the same stochastic dynamics of

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt$$

- Let $d = 1$, $(0, \infty)$, $1 \leq \alpha < \infty$, $\beta = 2$

$\Phi(x) = -\frac{\alpha}{2} \log x$ and $\Psi(x, y) = -\log |x - y|$.

- The distribution of N-particles are

$$\mu_{\text{bes},2}(d\mathrm{x}) = \frac{1}{\mathcal{Z}_\alpha} \prod_{j=1}^{N} x_j^\alpha \prod_{k<l}^{N} |x_k - x_l|^2 \prod_{m=1}^{N} dx_m. \tag{10}$$

- The associated SDEs with the Lagaurre ensemble are:

$$dX_t^{N,i} = dB_t^i + \{-\frac{1}{8N} + \frac{\alpha}{2X_t^{N,i}} + \sum_{j \neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}}\}dt \quad (1 \leq i \leq N). \tag{11}$$

- We set

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{\frac{\alpha}{2X_t^i} + \sum_{j \neq i}^{\infty} \frac{1}{X_t^i - X_t^j}\}dt \quad (i \in \mathbb{N}). \tag{12}$$

Here particles move in $(0, \infty)$.

- Bessel RPF $\mu_{\text{bes},2}$ is det RPF with

$$K_\alpha(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})\sqrt{y}J_\alpha(\sqrt{y})}{2(x-y)} \tag{13}$$

$$= \frac{\sqrt{x}J_{\alpha+1}(\sqrt{x})J_\alpha(\sqrt{y}) - J_\alpha(\sqrt{x})\sqrt{y}J_{\alpha+1}(\sqrt{y})}{2(x-y)}$$

- Extended Bessel kernel: $\mathbb{K}_\alpha(s,x;t,y), s,t \in \mathbb{R}^+, x,y \in \mathbb{R}^+$:

$$\mathbb{K}_\alpha(s,x;t,y) = \begin{cases} \int_0^1 du\, e^{-2u(s-t)} J_\alpha(2\sqrt{ux}) J_\alpha(2\sqrt{uy}) & \text{if} \quad s<t, \\ K_\alpha(x,y) & \text{if} \quad s=t, \\ -\int_1^\infty du\, e^{-2u(s-t)} J_\alpha(2\sqrt{ux}) J_\alpha(2\sqrt{uy}) & \text{if} \quad s>t. \end{cases}$$

- ISDE have unique strong solutions, and the associated dynamics are same as those of given by $\mathbb{K}_\alpha$.
- N-particles converge the infinte volume dynamics.

General theories of ISDEs of interacting Brownian motions: (1) Quasi-Gibbs, (2) Log derivative.

- For $\sigma : \mathbb{R}^d \times \mathsf{S} \to \mathbb{R}^{d^2} \cup \{\infty\}$, $b : \mathbb{R}^d \times \mathsf{S} \to \mathbb{R}^d \cup \{\infty\}$

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{\diamond i})dB_t^i + b(X_t^i, \mathsf{X}_t^{\diamond i})dt \quad (i \in \mathbb{N}) \tag{IBM}$$

Here $\mathsf{X}_t^{\diamond i}$ denotes the unlabeled particles other than X_t^i:

$$\mathsf{X}_t^{\diamond i} = \sum_{j \neq i} \delta_{X_t^j}$$

- Differential equation of μ:

$$2b(x, \mathsf{y}) = \{\nabla_x a(x, \mathsf{y}) + a(x, \mathsf{y}) \nabla_x \log \mu^{[1]}\} \tag{GD}$$

Here $a(x, \mathsf{y}) = \sigma(x, \mathsf{y})^t \sigma(x, \mathsf{y})$, $\mathsf{d}_\mu = \nabla \log \mu^{[1]}$ is log derivative.

- 1st Theorey (O.):

(GD) has a quasi-Gibbsian solution $\mu \Rightarrow$ (IBM) have solutions.

- 2nd Theory (O.-Tanemura):

Solutions of (IBM) are strong solutions and pathwise unique.

- $S_r = \{s \in \mathbb{R}^d; |s| \le r\}$, $\quad \mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i},\ \mathsf{s}(S_r) < \infty (\forall r)\}$

Def: μ is Ψ-quasi-Gibbs measure if $\exists\ c_{r,\xi}^m$ s.t.

$$c_{r,\xi}^{m\ -1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m$$

Here Λ_r^m is the Poisson RPF with intensity $1_{S_r} ds$ on $\{\mathsf{s}(S_r) = m\}$.

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi)).$$

Here $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$.

- μ_x: reduced Palm m. cond. at x $\mu^{[1]}$: 1-Campbell m. on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^{[1]}(A \times B) = \int_{A \times B} \rho^1(x) \mu_x(d\mathsf{s}) dx$$

Def: $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu^{[1]})$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}^\mu d\mu^{[1]} = - \int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^{[1]} \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_0$$

Strong solutions of ISDEs: Remarks 1/2

Remark: Short history of strong solutions of the ISDEs

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{i=1,\, j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt$$

- Lang ('78, '79, ZWVG) : $\Psi \in C_0^3(\mathbb{R}^d)$, μ is grand canonical Gibbs m. "stationary Markov solutions". Lippner, Rost (1D)
- Fritz ('87, AOP): $\Psi \in C_0^3(\mathbb{R}^d)$, μ is grand canonical Gibbs m. $d \leq 4$, "Non-equilibrium solutions" and "strong Markov".
- Tamemura ('96 PTRF):Hard core balls, "strong Markov"
- Rölly-Tanemura: hard core + "exponential decay Ψ"
- Tsai has proved Dyson for all $\beta > 0$. Non-equilibrium.

- All authors but Tsai above used "Ito scheme" in infinite-dimension space. i.e., they use the "Lipschitz continuity" of the coefficients. This makes their proof very hard.

Gibbs measures with Ruelle's class potentials Ψ: For $i \in \mathbb{N}$

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \neq i}^{\infty} \nabla \Psi (X_t^i - X_t^j) dt. \tag{14}$$

Lennard-Jones 6-12 potential: Set $d = 3$, $0 < \beta$.

$$\Psi_{6,12}(x) = |x|^{-12} - |x|^{-6}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}} \} dt \ .$$

Riesz potentials of Ruelle's class: Let $d < a \in \mathbb{N}$, $0 < \beta$.

$$\Psi_a(x) = (\beta / a) |x|^{-a}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt. \tag{15}$$

Ginibre RPF: $d = 2$; $\mathbb{R}^2 = \mathbb{C}$.

- Dist of eigenvalues of non-Hermitian Gaussian RM are:

$$\mu^N_{\mathrm{gin},2}(ds) = \frac{1}{\mathcal{Z}_N} \prod_{i=1,\, i<j}^{N} |s_i - s_j|^2 \prod_{k=1}^{N} \frac{1}{\pi} e^{-|s_k|^2} d\Lambda_N$$

- $\Phi(x) = x^2$, $\Psi(x) = -2\log|x|$.
- $\mu_{\mathrm{gin},2}$ is the TD limit of

$$\mu_{\mathrm{gin},2} = \lim_{N\to\infty} \mu^N_{\mathrm{gin},2}$$

- $\mu_{\mathrm{gin},2}$ is rotation and translation invariant.
- $\mu_{\mathrm{gin},2}$ is determinantal RPF with kernel

$$K_{\mathrm{gin},2}(x,y) = e^{-|x|^2/2} e^{x\bar{y}} e^{-|y|^2/2}.$$

That is,

$$\rho^n_{\mathrm{gin}}(\mathbf{x}_n) = \det[K_{\mathrm{gin},2}(x_i, x_j)]^n_{i,j=1}.$$

Examples: Ginibre RPF

- $d = 2$, $\Phi(x) = x^2$, $\Psi(x) = -2\log|x|$.

$$dX_t^i = dB_t^i - X_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{16}$$

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{17}$$

Thm 5 (O.12 PTRF, O.-Tanemura). *quad*

- *ISDE* (16) *has a pathwise unique, strong solution for* $\mu^{\ell}_{\text{gin},2}$*-a.s.*
- *ISDE* (17) *has a pathwise unique, strong solution for* $\mu^{\ell}_{\text{gin},2}$*-a.s.*
- *These solutions are the same.*

- This phenomena is a dynamical rigidity: Solutions trapped on the support of $\mu_{\text{gin},2}$, which is very thin subset and drift terms are equal on it.

Dynamical rigidity of Ginibre RPF 2

Ginibre RPF: $d = 2$, $\Psi(x) = -2\log|x|$.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{16}$$

Thm 6. *Each tagged particles are sub diffusive:*

$$\lim_{\varepsilon\to 0} \varepsilon X^i_{t/\varepsilon^2} = 0 \qquad \textit{for all } i \in \mathbb{N} \tag{18}$$

weakly in $C([0,\infty):\mathbb{R}^2)$ in $\mu_{\mathrm{gin},2}$-measure.

- Tagged particles are diffusive for Ruelle's class pot for $d \geq 2$. Let

$$dX_t^i = dB_t^i + \sum_{j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt.$$

If $d < a$, then $\lim_{\varepsilon\to 0} \varepsilon X^i_{t/\varepsilon^2}$ is non-degenerate.

- $\{\xi_k\}_{k=0}^{\infty}$:i.i.d. ξ_1: Gaussian on $\mathbb{C}$, mean 0, variance 1
- $F(z)$: planar GAF, i.e. an entire function s.t.

$$F(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k$$

- μ_{gaf} is the distribution of zero points of F
- μ_{gaf} is translation & rotation invariant.
- μ_{gaf} is different from, but similar to Ginibre RPF.
- μ_{gaf} has a rigidity more strict than Ginibre RPF.

Thm 7 (Peres-Ghosh).

$\mu_{\mathsf{gin}} \Rightarrow \mathsf{s}(S_r)$ *is deterministic* $\mu_{\mathsf{gin}}(\cdot|\pi_r^c(\xi))$*-a.s.*

$\mu_{\mathsf{gaf}} \Rightarrow \mathsf{s}(S_r)$ *&* $\sum_{s_i \in S_r} s_i$ *are deterministic* $\mu_{\mathsf{gaf}}(\cdot|\pi_r^c(\xi))$*-a.s.*

- Other rigidities of Ginibre RPF are known by O.-Shirai, Shirai.

$$\mathcal{H}_r(\mathsf{s}) = \sum_{s_i, s_j \in S_r, i<j} -2\log|s_i - s_j|$$

$$\Lambda_{r,m}^n = \Lambda_r(\cdot|\mathsf{s}(S_r) = n,\ \mathsf{m}_r(\mathsf{s}) = m)$$

μ_{gaf} is generalized $-2\log|x-y|$-quasi-Gibbs meaure

Thm 8 (Ghosh). *Let* $\xi_r^c = \pi_r^c(\xi)$.

$$c^{-1}(\xi_r^c)e^{-2\mathcal{H}_r(\mathsf{s})}\Lambda_{r,m}^n \le \mu_{\mathsf{gaf}}(\cdot|\xi_r^c) \le c(\xi_r^c)e^{-2\mathcal{H}_r(\mathsf{s})}\Lambda_{r,m}^n$$

Here $n = \mathsf{s}(S_r)$ *and* $m = \mathsf{m}_r(\mathsf{s})$ *are determined by* ξ_r^c *uniquely.*

Applying the 1st theory we have:

Thm 9 (O.).

μ_{gaf}-reversible diffusion $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ *exists.*

Thm 10 (Ghosh-O.-Shirai).

$\mathsf{d}_{\mu_{\text{gaf}}}$ *exists, and* $\mathbf{X} = (X_t^i)$ *satsfies ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_{\mu_{\text{gaf}}}(X_t^i, \mathsf{X}_t^{i\diamond})dt$$

- We do not obtained explicit representation of $\mathsf{d}_{\mu_{\text{gaf}}}$.

Hence we can not apply the 2nd theory for planar GAF at present.

- $\mathsf{d}_{\mu_{\text{gaf}}}$ is a limit of the log derivative $\mathsf{d}_{\mu_{\text{gaf}}}^N$ of μ_{gaf}^N.

Here μ_{gaf}^N is the dist of zero points of GAF F^N with order N.

$$F^N(z) = \sum_{k=0}^{N} \frac{\xi_k}{\sqrt{k!}} z^k$$

Geometric rigidity of planar GAF yields dynamical rigidity.

The proof for Ginibre RPF is still varid for GAF.

Thm 11.

Let X_t^i be a tagged particle of GAF. Then

$$\lim_{\epsilon \to \infty} \epsilon X_{t/\epsilon^2} = 0$$

Conj: GAF has dynamical rigidity strickly more rigid than that of Ginibre RPF.